Topological quantum order: stability under local perturbations

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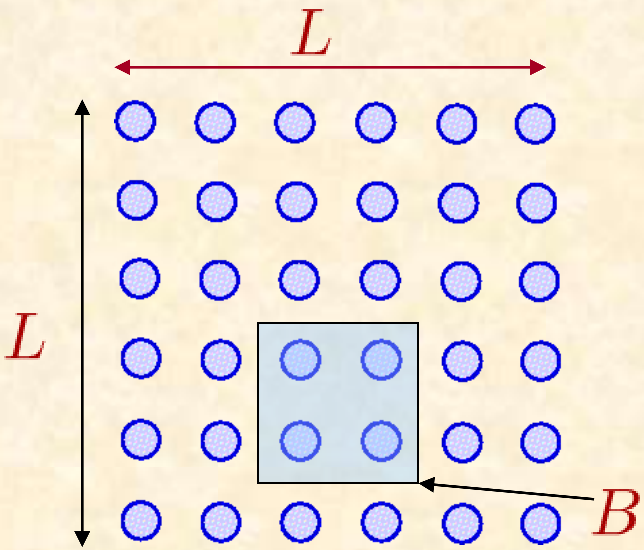
Why the existence of topologically ordered phases of matter is surprising?

1. Ground states of TQO models are highly entangled. This entanglement cannot be accounted only by local correlations. Non-local entanglement in macroscopic systems is extremely fragile.

2. How can nature prepare these highly entangled states without having a large-scale quantum computer?

3. Many models of TQO require multi-spin interactions which are not very realistic.

Quantum spin lattices



Lattice Λ

Finite-dimensional quantum spins live at sites.

Hamiltonian of the ideal model:

$$H_0 = \sum_{B \subseteq \Lambda} Q_B$$

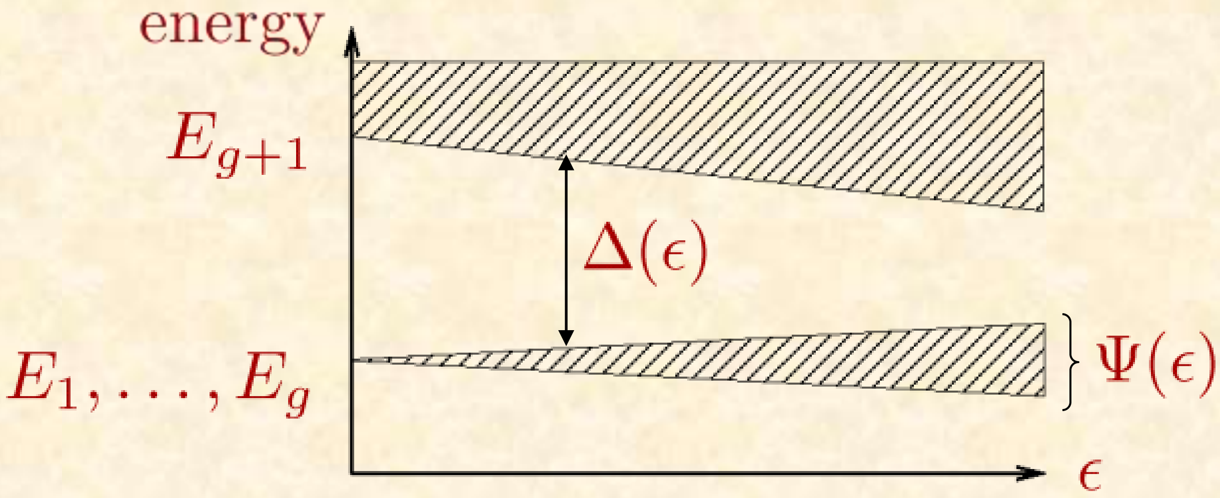
$$\|Q_B\| \le 1$$

To what extent ground state properties of H_0 are sensitive to addition of weak local perturbations?

$$H_0 \to H_0 + \epsilon V, \qquad V = \sum_{B \subseteq \Lambda} V_B, \qquad \|V_B\| \le 1.$$

Gap stability

The ground state $\Psi(0)$ is a g-dimensional subspace. The ground subspace $\Psi(\epsilon)$ includes g smallest eigenvalues of $H_0 + \epsilon V$. Well-defined as long as $\Delta(\epsilon) > 0$.



Main goal: find sufficient conditions under which H_0 has a non-zero stability radius ϵ_0, that is, the gap $\Delta(\epsilon)$ has a constant (L-independent) lower bound on the interval $\epsilon \in [0, \epsilon_0]$ for some $\epsilon_0 > 0$.

Exact quasi-adiabatic continuation theorem (Hastings 2005,2010, Osborne 2007)

Suppose the spectral gap $\Delta(\lambda)$ has a constant lower bound for $\lambda \in [0, \epsilon]$. Then $\Psi(0)$ and $\Psi(\epsilon)$ can be mapped to each other by some unitary operator U,

$$\Psi(\epsilon) = U \cdot \Psi(0)$$

and U can be implemented in time $t = O(1)$ as $L \to \infty$.

$$U = T \cdot \exp\left(i \int_0^t ds\, G_\epsilon(s) \right)$$

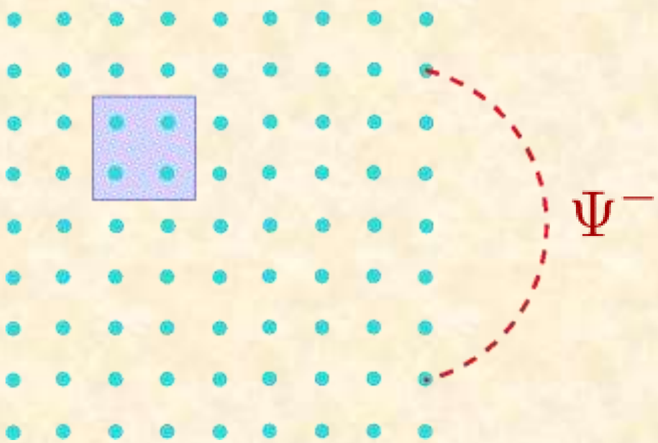
Here $G_\epsilon(s)$ is a local (approximately) Hamiltonian with bounded strength of interactions.

The states $\Psi(0)$ and $\Psi(\epsilon)$ are in the same "topological phase" if $\Psi(\epsilon) = U \cdot \Psi(0)$ and U can be implemented by evolution under a local Hamiltonian in time $O(1)$.

Can it happen that all ground states of local Hamiltonians are in the same phase?

Topological trivial phase = product states

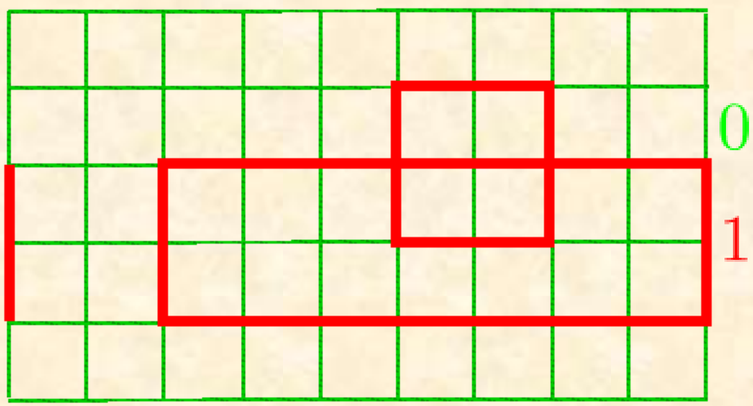
States with long-range bipartite entanglement would be in a non-trivial phase



Generating a singlet between remote qubits by local unitary dynamics can take time of order L (Lieb-Robinson bound).

However, such states cannot appear as ground states of local Hamiltonians.

Topological quantum order: a pattern of long-range multi-partite entanglement that can be present in the ground states of local Hamiltonians.



$|C\rangle$

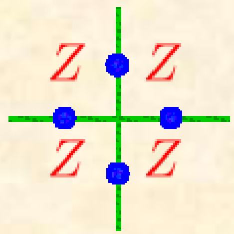
Toric code state (Kitaev 97)

Qubits live on links.

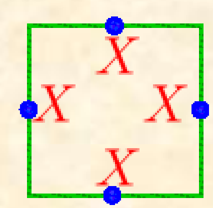
$$|\Psi_{tc}\rangle \sim \sum_{\text{cycles}} |C\rangle$$

$$H_0 = -\sum_{\text{stars}} A_s - \sum_{\text{plaquettes}} B_p$$

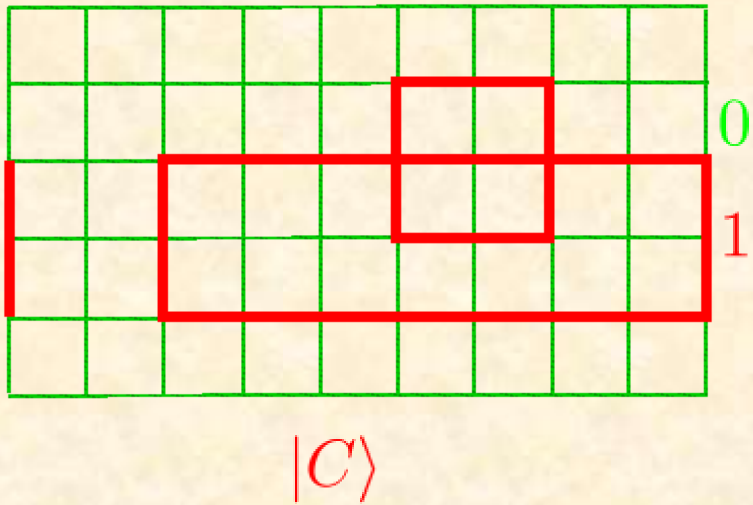
Star operators:

$A_s =$ 

Plaquette operators:

$B_p =$ 

Topological quantum order: a pattern of long-range multi-partite entanglement that can be present in the ground states of local Hamiltonians.



Toric code state (Kitaev 97)

Qubits live on links.

$$|\Psi_{tc}\rangle \sim \sum_{\text{cycles}} |C\rangle$$

Theorem (S.B., Hastings, and Verstraete 2006):
The toric code state on a lattice of size L cannot be generated from the product state by local unitary dynamics in time $o(L)$.

$\Rightarrow$ Ψ_{tc} and the product state are in different phases.

Previous work: gap stability for the toric code for several special perturbations:

Trebst, Werner, Troyer, Shtengel, Nayak (2007)
Magnetic field diagonal in the Z-basis
Reduction to the 2D transverse field Ising model

Vidal, Thomale, Schmidt, Dusuel (2009)
Magnetic field diagonal in the Y-basis
Reduction to the Xu-Moore model; use of self-duality

Klich (2009)
Generic perturbations; non-degenerate ground state
Generic perturbations diagonal in the Z-basis
Cluster expansions for the partition function

New results: sufficient gap stability conditions for a large class of ideal models H_0 and generic perturbations.

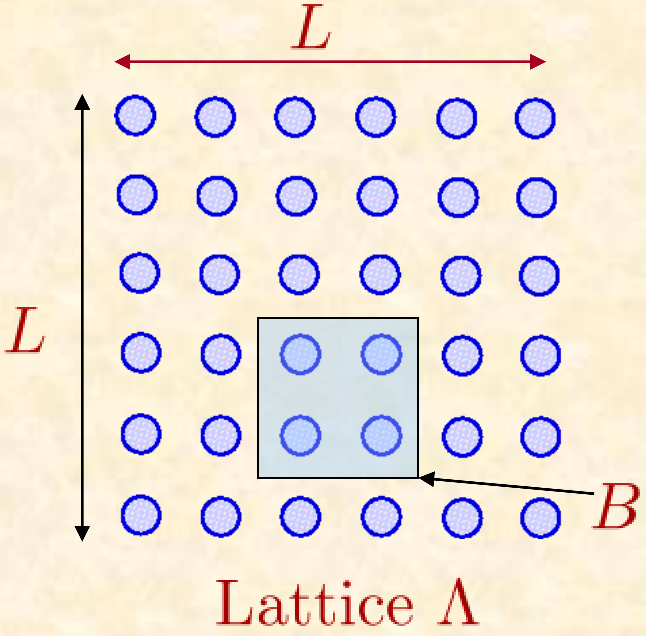
TQO-1: Ground subspace of H_0 is a quantum code with a macroscopic distance.

TQO-2: Consistency between the global and the local ground subspaces of H_0
(formal definition will appear later)

TQO-1 is only a property of the ground state

TQO-2 is a property of the Hamiltonian

We shall prove that TQO-1,2 together are sufficient for stability under generic local perturbations



Unperturbed Hamiltonian:

$$H_0 = \sum_{B \subseteq \Lambda} Q_B$$

Q_B is a Hermitian operator acting only on a cluster B

Only 2×2 clusters

Q_B must be a projector: $Q_B^2 = Q_B$

Projectors must pairwise commute: $Q_B Q_C = Q_C Q_B$

Ground states of H_0 are zero-eigenvectors of every projector Q_B

To summarize, we need three properties of the ideal model:

- Spatially local
- Frustration free
- Term-wise commuting

Several extra conditions related to TQO will be introduced later...

Examples:

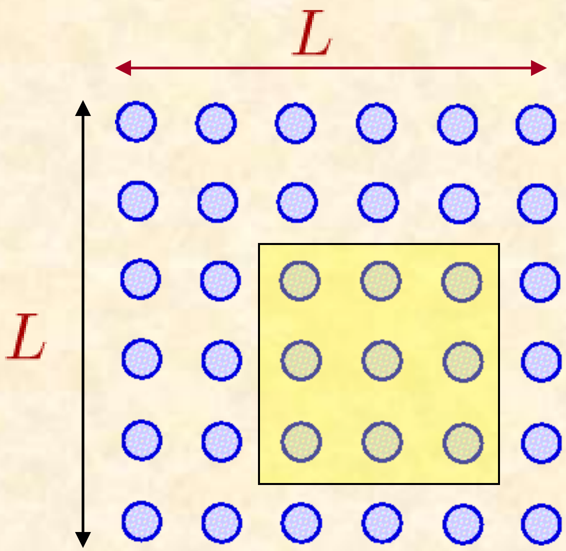
- The toric codes and the surface codes
- Topological color codes
- Quantum double models
- String-net models

$$H_0 = \sum_B Q_B$$

- Any of the above models with excitations

Kitaev 97
Bombin and Martin-Delgado 06
Levin and Wen 05

Commutativity guarantees that H_0 has constant spectral gap ($\Delta \geq 1$) above the ground state !



Lattice Λ

Generic perturbations:

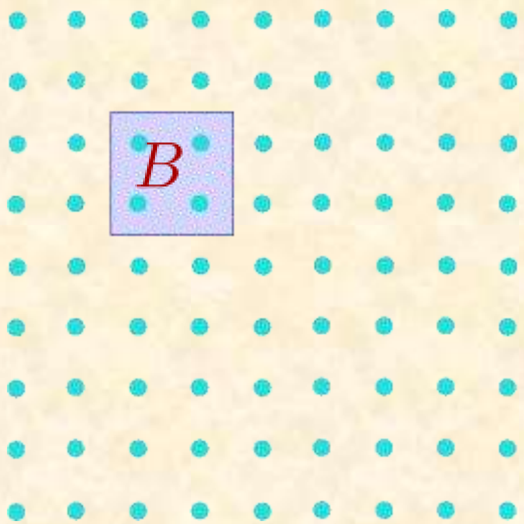
$$V = \sum_{B \subseteq \Lambda} V_B$$

V_B is a Hermitian operator acting only on a cluster B

Exponential decay of interactions:

For clusters of size $r \times r$ $\quad \max_B \|V_{r,B}\| \leq \exp(-\mu r)$

$\mu =$ decay rate

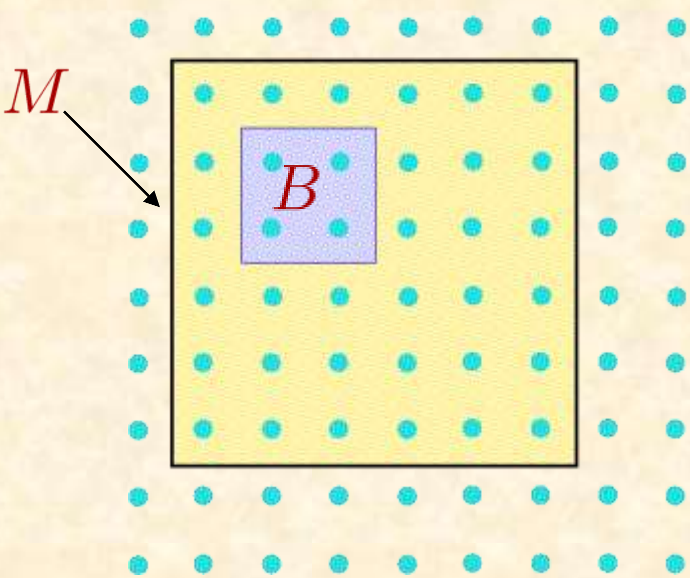


$$H_0 = \sum_B Q_B$$

Global ground subspace:

$$P = \{|\psi\rangle \,:\, Q_B\,|\psi\rangle = 0 \quad \forall B\}$$

$$P \equiv \Psi(0)$$

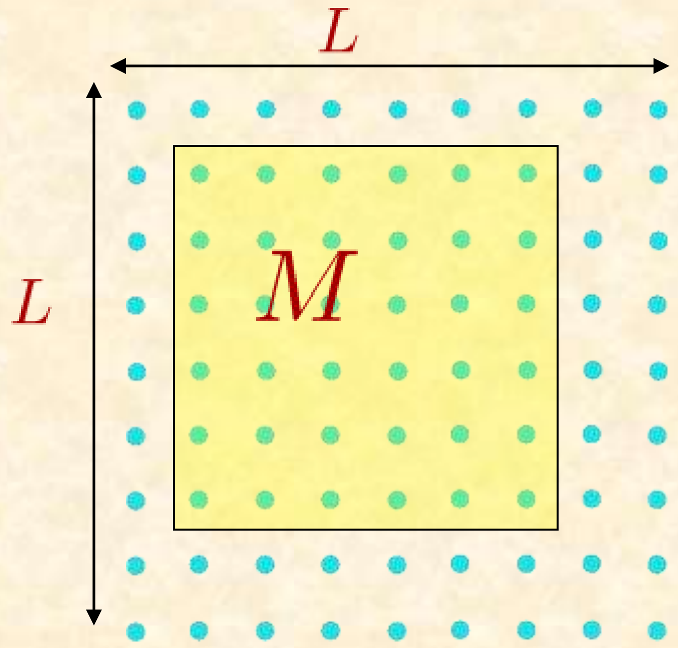


Local ground subspace:

$$P_M = \{|\psi\rangle \,:\, Q_B\,|\psi\rangle = 0 \quad \forall B \subseteq M\}$$

zero eigenvectors only for projectors Q_B whose support is contained in M.

TQO-1 (macroscopic distance):



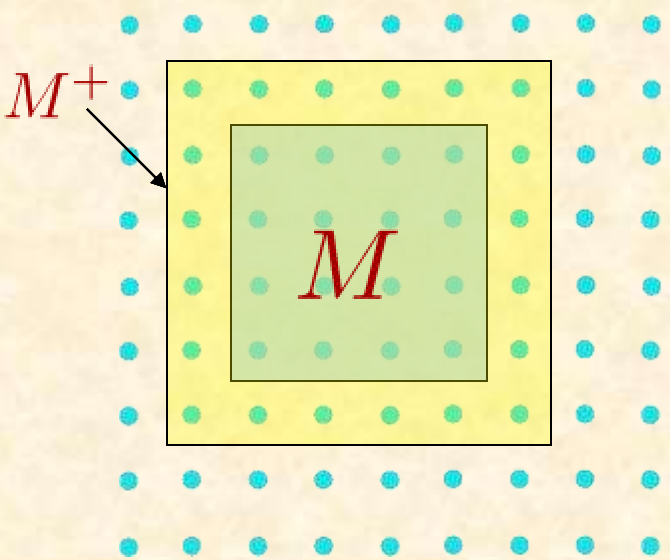
Global ground states cannot be distinguished locally:

$$\langle\psi|O_M|\psi\rangle = \langle\phi|O_M|\phi\rangle \quad \forall\, \psi, \phi \in P$$

for any operator O_M acting on M

Holds for all M's of size $\leq L^\alpha$.

TQO-2 (global-local consistency):

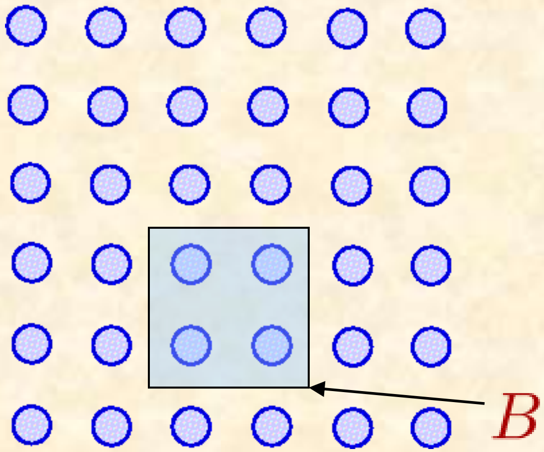


$$O_M\, P = 0 \quad \text{implies} \quad O_M\, P_{M^+} = 0$$

for any operator O_M acting on M

Holds for all M's of size $\leq L^\alpha$.

What is the meaning of TQO-2 for stabilizer codes?



Global ground states are invariant under the action of a stabilizer group

$$\mathcal{S} = \langle G_1, \ldots, G_m \rangle$$

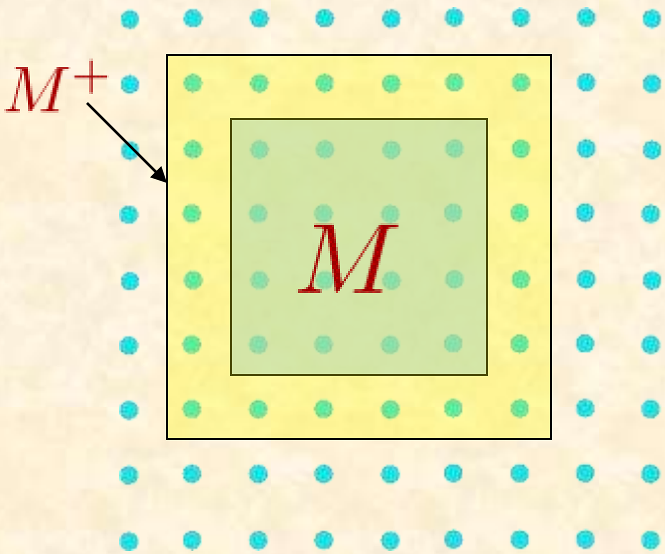
Generators G_a are pairwise commuting Pauli operators.

Each generator is supported on a 2×2 block B.

$$H_0 = \sum_B Q_B$$

Q_B penalizes states that violate at least one generator G_a supported inside B.

What is the meaning of TQO-2 for stabilizer codes?



TQO-2 (global-local consistency):

$O_M \, P = 0$ implies $O_M \, P_{M^+} = 0$

Holds for all M's of size $\le L^\alpha$.

Lemma. A stabilizer code Hamiltonian H_0 obeys TQO-2 iff any stabilizer $S \in \mathcal{S}$ supported on M can be written using only generators supported on M^+.

Toric code: if a loop operator has support inside M, it is a product of plaquette operators supported inside M.

To summarize, we need five properties of the ideal model H_0:

- Spatially local
- Frustration free
- Term-wise commuting
- Macroscopic distance (TQO-1)
- Local-global consistency (TQO-2)

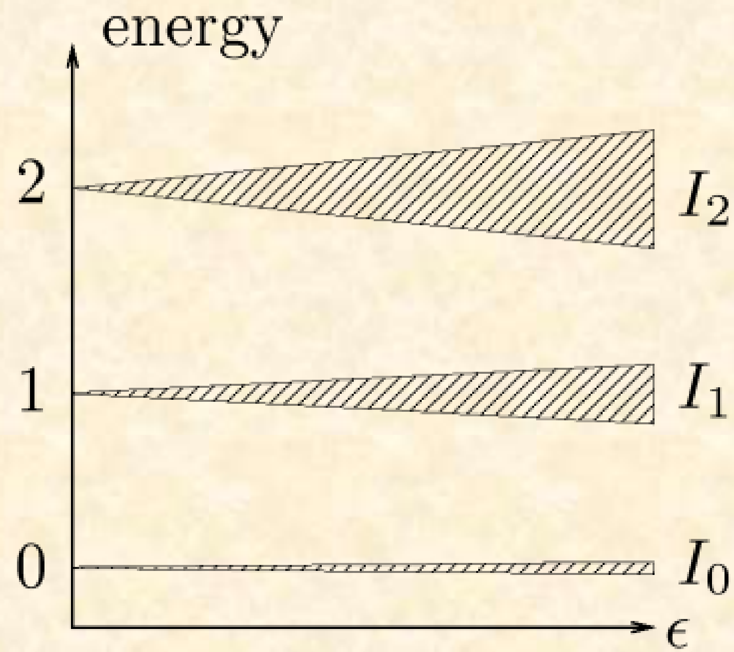
The perturbation V involves exponentially decaying interactions with strength J and decay rate $\mu > 0$.

Main theorem:

There exists a constant $c = c(\mu, \alpha)$ such that for all large enough L and for all $\epsilon > 0$ the spectrum of $H_0 + \epsilon V$ is contained (up to an overall energy shift) in the union of intervals

$$\bigcup_k I_k$$

- k runs over eigenvalues of H_0
- Interval I_k is centered at k
- $|I_k| = \epsilon c k$ for $k > 0$
- $|I_0| = \epsilon c \cdot exp(-\sqrt{L})$



Corollary: the spectral gap around I_k is at least $1/2$ for all $\epsilon \leq \epsilon_k = (1 + 2k)^{-1}(2c)^{-1}$.

1. The bound on the stability radius does not depend on the dimension of the local Hilbert spaces.

2. The overall energy shift may be a function of L and ϵ.

3. Conditions TQO-1,2 can be efficiently checked for any stabilizer code Hamiltonian.

4. The theorem applies to systems with symmetries. A system has a symmetry group $\mathcal{G}$ iff all local terms in H_0 and V commute with $\mathcal{G}$. Conditions TQO-1,2 must be obeyed only for operators O_M commuting with $\mathcal{G}$.

Symmetry protected topological order: non-trivial topological phases may exist even in 1D systems.
X. Chen, Z.-C. Gu, and X.-G. Wen
arXiv:0903.1069, arXiv:1008.3745

5. Conditions TQO-1,2 are well-defined for classical H_0.

TQO-1: All ground states are locally indistinhuishable. Hence unique ground state.

TQO-2: If some spin σ_u deviates from its ground state value, at least one interaction touching σ_u is violated.

Analogous to the Peierls condition in the stability theory for quantum perturbations of classical Hamiltonians (Datta, Frölich, Rey-Bellet 1997)

Why the ground state energy splitting is exp. small ?

Exact quasi-adiabatic continuation theorem implies

$$\psi_\alpha(1) = U \cdot \psi_\alpha(0), \quad \alpha = 1, \ldots, g,$$

where U describes unitary evolution under (approximately) local Hamiltonian for time $O(1)$.

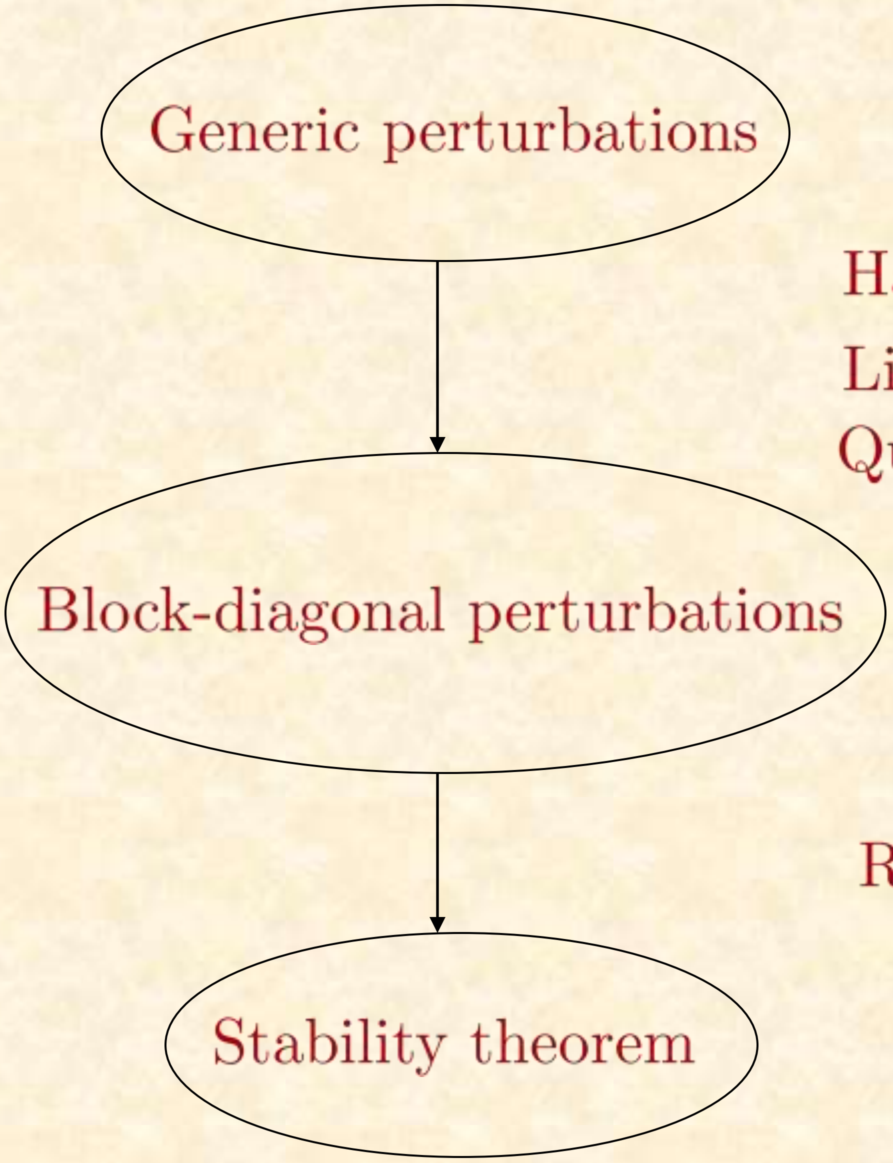
$$E_\alpha(1) = \langle \psi_\alpha(1) | H_0 + \epsilon V | \psi_\alpha(1) \rangle = \langle \psi_\alpha(0) | \tilde{H} | \psi_\alpha(0) \rangle,$$

$$\tilde{H} \equiv U^\dagger (H_0 + \epsilon V) U.$$

Lieb-Robinson bound implies that $\tilde{H}$ is a sum of (approximately) local interactions. Hence $\tilde{H}$ cannot distinguish orthogonal ground states $\psi_\alpha(0)$.

Hence all $E_\alpha(1)$ are (approximately) the same.

Sketch of the proof



Techniques:

Hamiltonian flow equations

Lieb-Robinson bounds

Quasi-adiabatic continuation

Relative bounds on V

Def. A perturbation V is relatively bounded by H_0 with a constant b iff

$$\|V\,\psi\| \leq b\,\|H_0\,\psi\|$$

for all vectors ψ

Lemma. The spectrum of $H_0 + V$ is contained in the union of intervals

$$I_k = [(1-b)k, (1+b)k]$$

where k runs over eigenvalues of H_0.

Applying the lemma to $H_0 + \epsilon V$ we get the desired energy bands I_k as long as $b \cdot |\epsilon| < 1$. Hence we need a bound $b = O(1)$.

Def. A perturbation V is called locally block-diagonal iff it is a sum of local operators preserving the global ground subspace of H_0, that is,

$$V = \sum_{B \subseteq \Lambda} V_B$$

$$V_B \cdot P \subseteq P$$

Macroscopic distance implies that V_B acts trivially on the ground states. Perform an overall energy shift to achieve

$$V_B \cdot P = 0$$

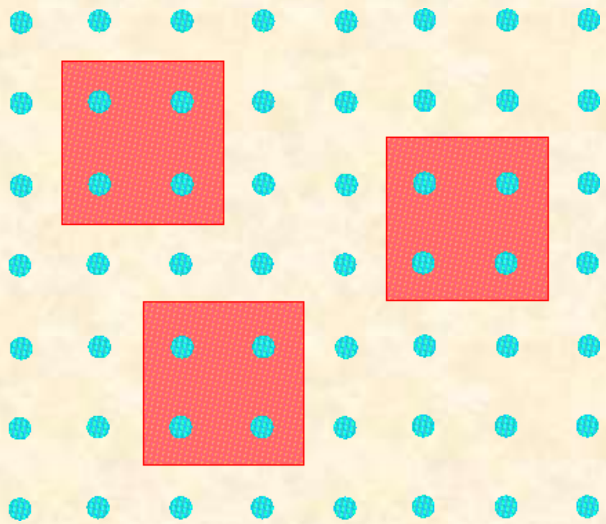
Lemma: A locally block-diagonal perturbation satisfying $V_B \cdot P = 0$ is relatively bounded by H_0 with a constant

$$b = b(\mu).$$

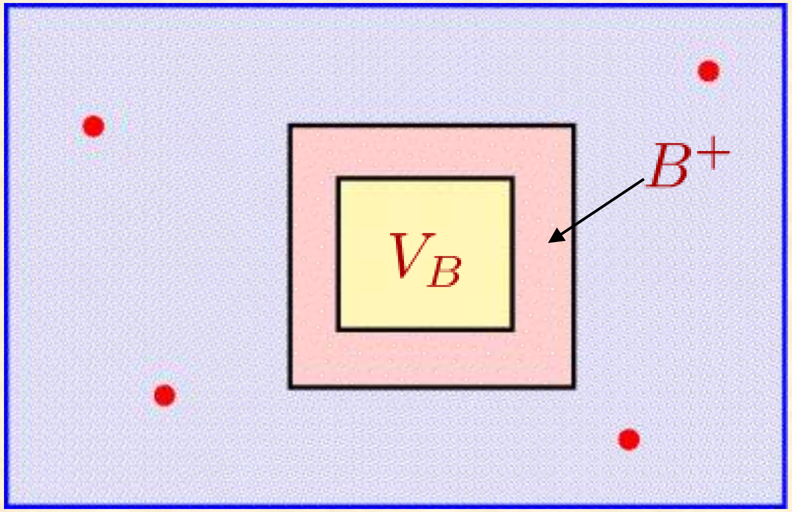
Block-diagonality $\Rightarrow$ relative boundness (rough idea)

Decompose the entire Hilbert space into sectors labeled by configurations of excitations.

Simplest case: excitation is a 2×2 square B such that $Q_B = 1$ instead of $Q_B = 0$.



Block-diagonality ⇒ relative boundness (rough idea)



$$V = \sum_{B \subseteq \Lambda} V_B, \quad V_B \cdot P = 0$$

TQO-2 implies $\quad V_B \cdot P_{B+} = 0$

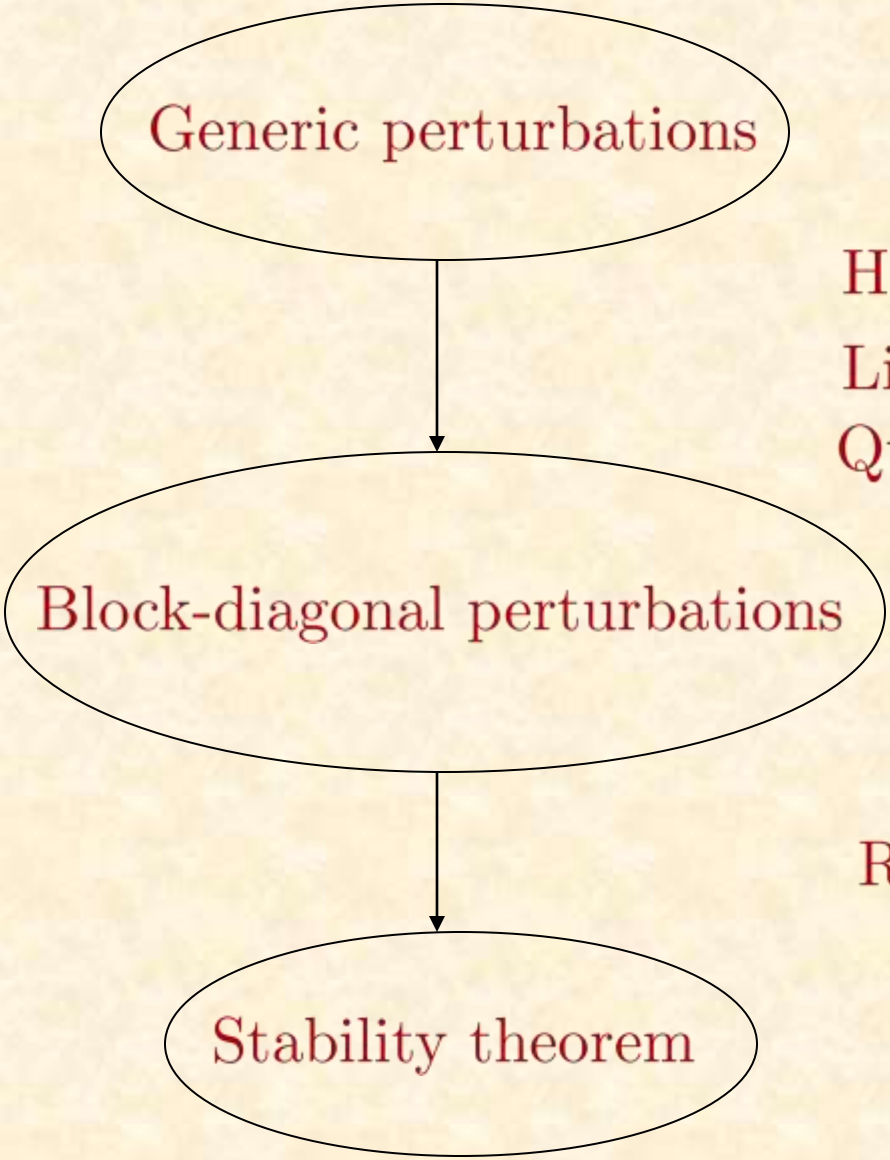
Hence V_B annihilates any sector that contains no excitations near B.

Assume for simplicty that ψ belongs to a sector with k excitations. Then there are only $O(k)$ terms V_B such that $V_B \psi \neq 0$.

$$\|V\,\psi\| \sim k \quad \text{and} \quad \|H_0\,\psi\| = k$$

Hence V is relatively bounded by H_0 with a constant b of order 1.

Sketch of the proof



Techniques:

Hamiltonian flow equations

Lieb-Robinson bounds

Quasi-adiabatic continuation

Relative bounds on V

Suppose V is not block-diagonal. We shall construct a unitary operator U such that

$$U(H_0 + \epsilon V)U^\dagger = H_0 + \epsilon W + H_{garbage}$$

W is a locally block-diagonal perturbation with a fast enough decay of interactions. W is relatively bounded by H_0 with a constant $b = O(1)$.

$H_{garbage}$ includes all unwanted terms. Must have exponentially small norm.

Hence ϵV changes eigenvalues of H_0 by a factor $1 \pm \epsilon b$ and an additive error $\|H_{garbage}\|$.

How to construct U (Hamiltonian flow equations):

First solve the linearized block-diagonalization problem. U only needs to make the Hamiltonian locally block-diagonal in the first order in ϵ:

$$U(H_0 + \epsilon V + \delta W)U^\dagger = H_0 + c\epsilon^2 V' + \delta' W' + H_{garbage}$$

Here W, W' are locally block-diagonal, V, V' are generic perturbations, c is some constant, and

$$\delta' \leq \delta + O(\epsilon)$$

$$U = \exp(\epsilon S), \quad S^\dagger = -S,$$

$$P^\perp \cdot ([S, H_0 + \delta W] + V) \cdot P = 0$$

Construct S using power series in δ. Use the Lieb-Robinson bound to show that V' and W'' decay fast enough.

How to construct U (Hamiltonian flow equations):

First solve the linearized block-diagonalization problem. U only needs to make the Hamiltonian locally block-diagonal in the first order in ϵ:

$$U(H_0 + \epsilon V + \delta W)U^\dagger = H_0 + c\epsilon^2 V' + \delta' W' + H_{garbage}$$

Here W, W' are locally block-diagonal, V, V' are generic perturbations, c is some constant, and

$$\delta' \leq \delta + O(\epsilon)$$

Iterate $m = O(\log L)$ times obtaining

$$\epsilon \to c\epsilon^2 \to c^3\epsilon^4 \to \ldots \to \frac{1}{c}(c\epsilon)^{2^m} = \exp(-L).$$

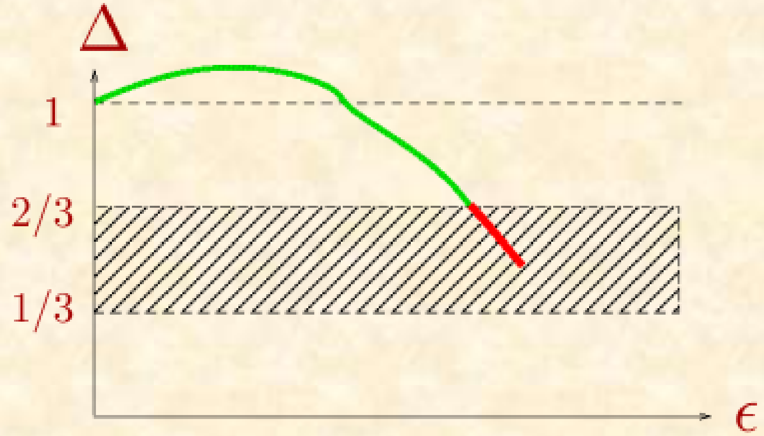
Include the residual V into $H_{garbage}$.

How to construct U (quasi-adiabatic continuation):

For any fixed H_0 and V the spectral gap $\Delta(\epsilon)$ of $H_0 + \epsilon V$ is a continuous function of ϵ such that $\Delta(0) \geq 1$.

We shall assume that the gap $\Delta(\lambda)$ is not too small on the interval $[0, \epsilon]$, say at least $1/3$, and use this assumption to show that that the gap $\Delta(\epsilon)$ is much larger than $1/3$, say, $\Delta(\epsilon) \geq 2/3$.

If this holds for all $\epsilon \in [0, \epsilon_0]$ then $\Delta(\epsilon) \notin [1/3, 2/3]$ for any $\epsilon \in [0, \epsilon_0]$. By continuity it implies $\Delta(\epsilon) \geq 2/3$ for all $\epsilon \in [0, \epsilon_0]$.



How to construct U (quasi-adiabatic continuation):

Now we are in the settings of the exact quasi-adiabatic evolution theorem:

$$\Psi(\epsilon) = U \cdot \Psi(0) \equiv U \cdot P,$$

where U describes unitary evolution under (approximately) local Hamiltonian for time $O(1)$.

The Hamiltonian

$$\tilde{H} = U^\dagger (H_0 + \epsilon V) U$$

is globally block-diagonal, that is, $\tilde{H} \cdot P \subseteq P$.

This is almost what we need:

$$\tilde{H} = H_0 + W, \quad W = U^\dagger H_0 U - H_0 + \epsilon U^\dagger V U$$

W has strength $O(\epsilon)$ and fast enought decay of interactions. However W is only globally block-diagonal, $W \cdot P \subseteq P$.

How to construct U (quasi-adiabatic continuation):

Remaining step: reduction from global to local block-diagonality (the hard part)

$$\tilde{H} = \int_{-\infty}^{\infty} g(t)\, e^{i\tilde{H}t}\, \tilde{H} e^{-i\tilde{H}t}$$

Using the assumption that $\tilde{H}$ is gapped one can choose the filter function $g(t)$ such that

$$W = \sum_{u\in\Lambda} W_u, \qquad W_u \cdot P \subseteq P, \qquad W_u = \sum_{B\ni u} W_B,$$

The magnitude of W_B decays fast enough for large clusters B. However individual terms W_B do not preserve P.

One extra trick is needed to show that W_u can be approximated by a locally block-diagonal operator. This approximation relies on TQO-1,2.

Conclusions

(1) The spectral gap of spin Hamiltonians composed of local commuting projectors and satisfiying conditions TQO-1,2 does not close in a presence of generic local perturbations.

(2) Conditions TQO-1,2 can be extended to systems with symmetries.

(3) Lieb-Robinson bound and the quasi-adiabatic continuation permit analysis of perturbed quantum systems which does not rely on perturbative expansions.