Decoupling:
A building block for quantum information theory

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Information (Shannon) theory

- A practical question:
  - How to best make use of a given communications resource?
- A mathematico-epistemological question:
  - How to quantify uncertainty and information?
- Shannon:
  - Solved the first by considering the second.
  - ~~A~~ *mathematical* theory of communication [1948]

The

Shannon theory provides

- Practically speaking:
  - A holy grail for error-correcting codes
- Conceptually speaking:
  - A operationally-motivated way of thinking about correlations
- What's missing (for a quantum mechanic)?
  - Features from linear structure: *Entanglement* and *non-orthogonality*

Quantum Shannon Theory provides

- General theory of interconvertibility between different types of communications resources: qubits, cbits, ebits, cobits, sbits…
- Relies on a
  - Major simplifying assumption:
    Computation is free
  - Minor simplifying assumption:
    Noise and data have regular structure

These lectures

- You will learn one, very powerful result
- Its various names:
  - State transfer
  - Fully quantum Slepian-Wolf
  - The mother of all protocols
- Part I: applications
- Part II: the proof

VON NEUMANN ENTROPY:

$$H(A)_\rho := H(\rho_A) = -\operatorname{tr} \rho_A \log \rho_A$$

$$0 \leq H(A)_\rho \leq \log \dim A$$

$$\Updownarrow \qquad\qquad \Updownarrow$$

ρ PURE $\qquad$ ρ MAX MIX

MUTUAL INFORMATION:

$$I(A;B)_\rho := H(A)_\rho + H(B)_\rho - H(AB)_\rho$$

$$0 \leq I(A;B)_\rho \leq 2 \log \dim A$$

$$\Updownarrow \qquad\qquad \Updownarrow$$

ρ PRODUCT: $\rho_{AB} = \rho_A \otimes \rho_B$ $\qquad$ ρ MAX ENTANGLED

CONDITIONAL ENTROPY & COHERENT INFO

$$H(A|B)_\rho := H(AB)_\rho - H(B)_\rho =: -I(A\rangle B)_\rho$$

$$-\log \dim A \leq H(A|B)_\rho \leq \log \dim A$$

BASIC RESOURCES: SINGLE SENDER & SINGLE RECEIVER

ALICE — BOB

[c → c]: ONE USE OF AN ALICE → BOB NOISELESS CBIT CHANNEL ($\overline{id}_2$)

[q → q]: ONE USE OF AN ALICE → BOB NOISELESS QUBIT CHANNEL (id_2)

[qq]: ONE ALICE-BOB EBIT $\frac{1}{\sqrt{2}}(|0\rangle|0\rangle + |1\rangle|1\rangle)$

RESOURCE INEQUALITIES: A SHORTHAND

OBVIOUS: $[q \to q] \geq [c \to c]$

LHS CAN SIMULATE RHS

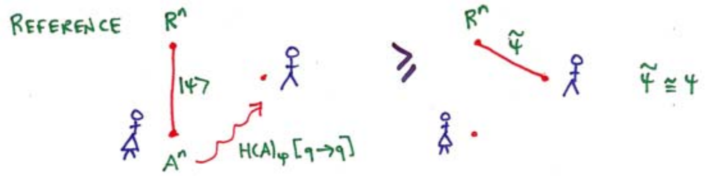
ENTANGLEMENT DISTRIBUTION: $[q \to q] \geq [qq]$

SUPERDENSE CODING: $[q \to q] + [qq] \geq 2[c \to c]$

OF USES

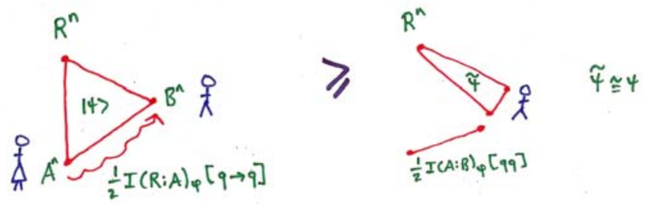
TELEPORTATION: $[qq] + 2[c \to c] \geq [q \to q]$

SCHUMACHER COMPRESSION: $|\psi\rangle_{R^nA^n} = (|\psi\rangle_{RA})^{\otimes n}$



$\tilde{\psi} \cong \psi$

OUR MONDAY GOAL: STATE TRANSFER: $|\psi\rangle_{R^nA^nB^n} = (|\psi\rangle_{RAB})^{\otimes n}$



$\tilde{\psi} \cong \psi$

NOTE: STATE TRANSFER GENERALIZES SCHUMACHER COMPRESSION

IF $|\varphi\rangle_{RA}$ IS PURE, THEN

$$\frac{1}{2} I(R;A)_\varphi = \frac{1}{2}[H(R)_\varphi + H(A)_\varphi - H(RA)_\varphi]$$

$$= \frac{1}{2}[H(A)_\varphi + H(A)_\varphi - 0] = H(A)_\varphi$$

RA PURE

$|\varphi\rangle_{RA}$ PURE $\Rightarrow \varphi_R$ AND φ_A HAVE SAME NON-ZERO EIGENVALUES

RESOURCE INEQUALITY FROM STATE TRANSFER

$$\langle \varphi_{AB} \rangle + \tfrac{1}{2} I(R:A)_\varphi [q \to q] \geq \tfrac{1}{2} I(A:B)_\varphi [qq] \qquad \text{(ST)}$$

EXTRACT E_n EBITS WHERE $\lim_{n\to\infty} E_n/n = \frac{1}{2} I(A:B)_\varphi$

ALLOW Q_n USES OF id_2 WHERE $\lim_{n\to\infty} Q_n/n = \frac{1}{2} I(R:A)_\varphi$

ALLOW ALICE AND BOB MANY USES $(n \to \infty)$ OF MIXED STATE φ_{AB}. ie $\varphi_{AB}^{\otimes n}$

SMALL IMPERFECTIONS ALLOWED BUT MUST VANISH AS $n \to \infty$.

ENTANGLEMENT DISTILLATION

GOAL: $\langle \varphi_{AB} \rangle + C[c \to c] \geq E[qq]$

WANT TO REPLACE $[q \to q]$ IN (ST) BY $[c \to c]$. HOW?

TELEPORTATION: $[qq] + 2[c \to c] \geq [q \to q] \qquad \text{(TP)}$

SUBSTITUTE (TP) INTO (ST):

$$\langle \varphi_{AB} \rangle + \tfrac{1}{2} I(R:A)_\varphi \left\{ [qq] + 2[c \to c] \right\} \geq \tfrac{1}{2} I(A:B)_\varphi [qq]$$

$$\langle \varphi_{AB} \rangle + I(R:A)_\varphi [c \to c] \geq \left\{ \tfrac{1}{2} I(A:B)_\varphi - \tfrac{1}{2} I(R:A)_\varphi \right\} [qq]$$

$$\boxed{\langle \varphi_{AB} \rangle + I(R:A)_\varphi [c \to c] \geq I(A \rangle B)_\varphi [qq]}$$

HASHING INEQUALITY

CHECK: IS $\frac{1}{2} I(A:B)_\varphi - \frac{1}{2} I(R:A)_\varphi = I(A \rangle B)_\varphi$ FOR $|\psi\rangle_{RAB}$?

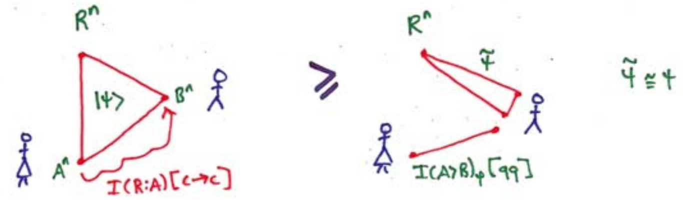
$$\tfrac{1}{2} I(A:B)_\varphi - \tfrac{1}{2} I(R:A)_\varphi = \tfrac{1}{2} \left\{ H(A)_\varphi + H(B)_\varphi - H(AB)_\varphi - H(R)_\varphi - H(A)_\varphi + H(RA)_\varphi \right\}$$

$$= \tfrac{1}{2} \left\{ H(B)_\varphi - H(AB)_\varphi - H(AB)_\varphi + H(B)_\varphi \right\}$$

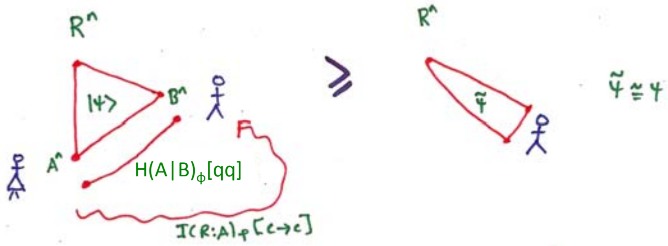
$$= H(B)_\varphi - H(AB)_\varphi$$

$$= I(A \rangle B)_\varphi$$

HAVE ACTUALLY SHOWN, FOR $|\psi\rangle_{R^n A^n B^n} = (|\psi\rangle_{RAB})^{\otimes n}$:



OR, EQUIVALENTLY,



NOTE: $H(A|B) \gtrless 0$ ARE BOTH POSSIBLE!

>0: ENTANGLEMENT IS CONSUMED

<0: ENTANGLEMENT IS GENERATED

INTERPRETATION OF $H(A|B)_\varphi$:

(QUANTUM) UNCERTAINTY ABOUT A WHEN B IS GIVEN.

<0: MORE THAN CERTAIN

ANOTHER APPLICATION: NOISY SUPERDENSE CODING

EXPLOIT MIXED φ_{AB} TO ENHANCE CLASSICAL COMMUNICATION

(ST) $\langle \varphi_{AB} \rangle + \frac{1}{2} I(R:A)_\varphi [q\to q] \geq \frac{1}{2} I(A:B)[qq]$

WANT $[c\to c]$ ON RHS. HOW?

USE SUPERDENSE CODING:

(SDC) $[qq] + [q\to q] \geq 2[c\to c]$

$$[qq] \geq 2[c\to c] - [q\to q]$$

SUBSTITUTE (SDC) INTO (ST):

$$\langle \varphi_{AB} \rangle + \frac{1}{2} I(R:A)_\varphi [q\to q] \geq \frac{1}{2} I(A:B)_\varphi \left\{ 2[c\to c] - [q\to q] \right\}$$

$$\langle \varphi_{AB} \rangle + \left\{ \frac{1}{2} I(R:A)_\varphi + \frac{1}{2} I(A:B)_\varphi \right\} [q\to q] \geq I(A:B)_\varphi [c\to c]$$

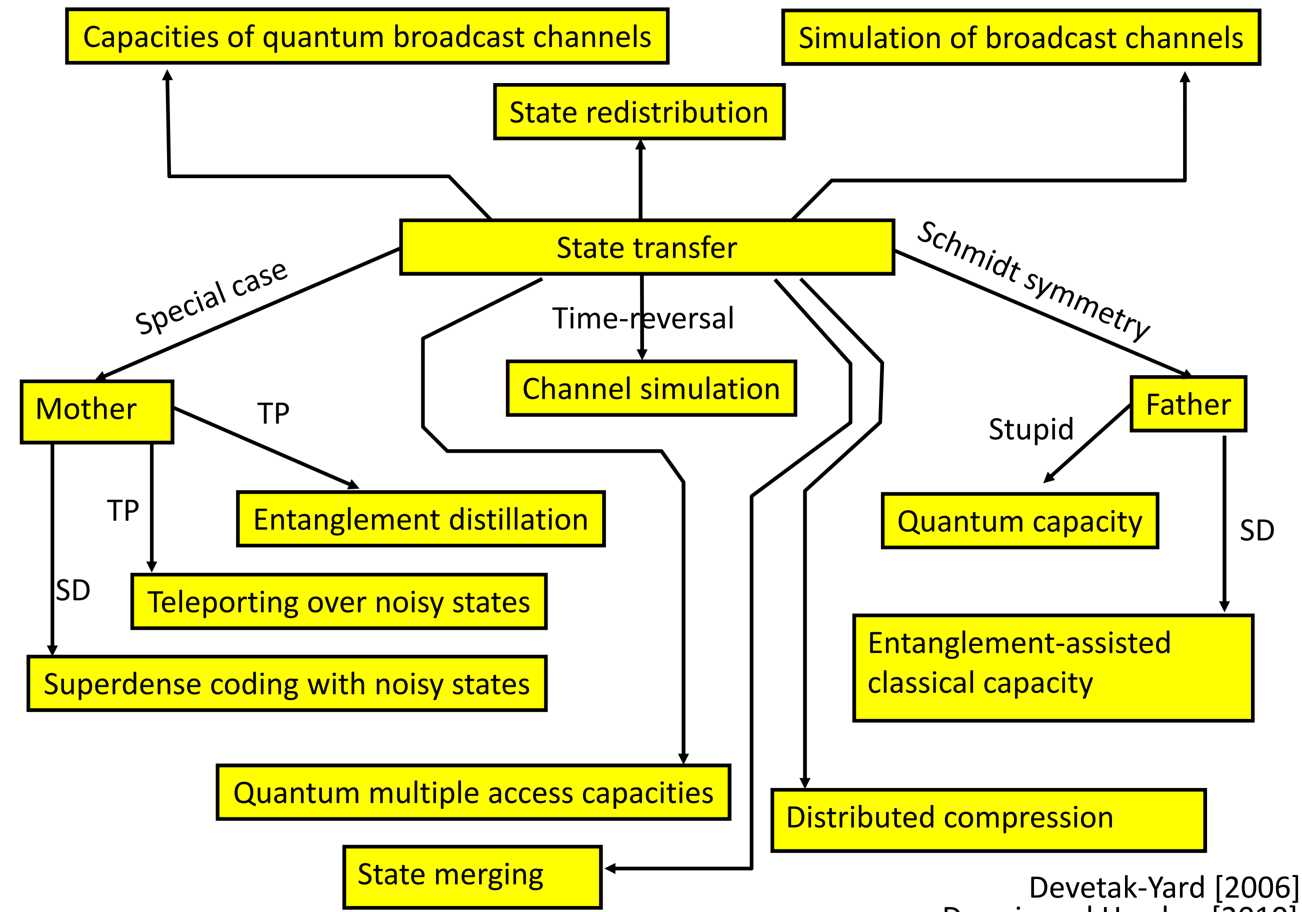
(SDC++) $\boxed{\langle \varphi_{AB} \rangle + H(A)_\varphi [q\to q] \geq I(A:B)_\varphi [c\to c]}$

CHECK: $\frac{1}{2} I(R:A)_\varphi + \frac{1}{2} I(A:B)_\varphi$

$$= \frac{1}{2} \left\{ H(R)_\varphi + H(A)_\varphi - H(RA)_\varphi + H(A)_\varphi + H(B)_\varphi - H(AB)_\varphi \right\}$$

$$= \frac{1}{2} \left\{ H(R)_\varphi + H(A)_\varphi - H(RA)_\varphi + H(A)_\varphi + H(RA)_\varphi - H(R)_\varphi \right\}$$

$$= H(A)_\varphi$$



Devetak-Yard [2006]
Dupuis and Hayden [2010]
Abeyesinghe, Devetak, Hayden, Winter [2006]

Devetak, Harrow, Winter [2003]

Part II:
Proof of the state transfer theorem

(The easy route to quantum information guru status)

MIXED STATE DISTANCE MEASURES

1) FIDELITY (GENERALIZES INNER PRODUCT)

$$F(\rho, \sigma) := \max_{|\psi_\rho\rangle, |\psi_\sigma\rangle} |\langle \psi_\rho | \psi_\sigma \rangle|^2$$

OVER ALL PURIFICATIONS

$$= \max_{U_R} |\langle \psi_\rho | I \otimes U_R | \psi_\sigma \rangle|^2$$

FIXED PURIFICATIONS

OVER ALL UNITARIES ON PURIFYING SPACE

$$F(\rho, \sigma) = \begin{cases} 1 & \text{iff } \rho = \sigma \\ 0 & \text{iff } \rho \perp \sigma \end{cases}$$

2) TRACE DISTANCE

RECALL $\|X\|_1 := \mathrm{tr}\sqrt{X^\dagger X}$

IF $X = \sum_i \lambda_i |i\rangle\langle i|$ THEN $\|X\|_1 = \sum_i |\lambda_i|$

USE NORM TO MEASURE DISTANCE:

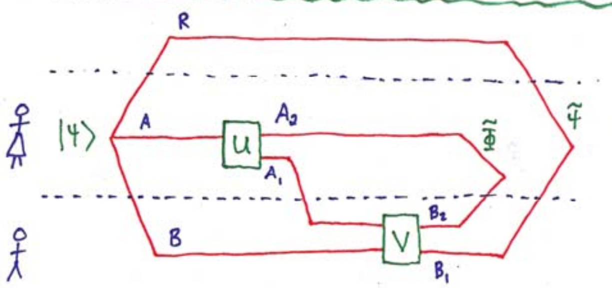
$$D(\rho, \sigma) := \|\rho - \sigma\|_1$$

(1) $\simeq$ (2): ROUGH EQUIVALENCE

$$1 - \sqrt{F} \le \frac{D}{2} \le \sqrt{1-F}$$

$$1 - D \le F \le 1 - \frac{D^2}{4}$$

STATE TRANSFER AS A CIRCUIT



$|\Phi\rangle$: MAX ENTANGLED

$\tilde{\Phi} \simeq \Phi$

$\tilde{\psi} \simeq \psi$

NEED TO FIND ISOMETRIC U, V ACCOMPLISHING THIS GOAL

WANT: $(V \circ U)|\psi\rangle_{RAB} \simeq |\psi\rangle_{RB_1} |\Phi\rangle_{A_2 B_2}$

THE DECOUPLING ARGUMENT

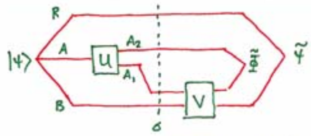
LET $|\omega\rangle = U|\psi\rangle_{RAB}$

SUPPOSE $1 - \epsilon < F(\psi_R \otimes \Phi_{A_2}, \omega_{RA_2}) \leftarrow (*)$

$$= \max_{V_{A_1 B}} {}_{RB_1}\langle\psi|_{A_2 B_2}\langle\Phi| I \otimes V_{A_1 B} |\omega\rangle$$

DEFINITION OF FIDELITY YIELDS V AUTOMATICALLY!

WILL SUFFICE TO DEMONSTRATE (*)!



$d_X := \dim X$

STRATEGY: SHOW THAT A "TYPICAL" U WILL DO THE JOB

TH'M [DECOUPLING]:
$$\int_{\mathcal{U}(A)} \left\| \sigma(U)_{RA_2} - \psi_R \otimes I_{A_2}/d_{A_2} \right\|_1^2 \, dU \le \frac{d_R\, d_A}{d_{A_1}^2} \mathrm{tr}\left[(\psi_{RA})^2 \right]$$

TO INTERPRET: RECALL SCHUMACHER COMPRESSION

LET Π_X BE THE TYPICAL PROJECTOR ON X FOR $\rho^{\otimes n}$

THEN: (i) $\mathrm{tr}\, \Pi_X \rho_X^{\otimes n} > 1-\epsilon$

(ii) $\mathrm{rank}\, \Pi_X \le 2^{n[H(X)_\rho + \epsilon]}$

(iii) $\Pi_X \rho_X^{\otimes n} \Pi_X \le 2^{-n[H(X)_\rho - \epsilon]} \Pi_X$

ie $\mathrm{Eig}(\Pi_X \rho_X^{\otimes n} \Pi_X) \le 2^{-n[H(X)_\rho - \epsilon]}$

SO:
$$\begin{aligned} \mathrm{tr}\left[(\Pi_X \rho_X^{\otimes n} \Pi_X)^2 \right] &\le \mathrm{rank}\, \Pi_X \cdot \left[\max \mathrm{Eig}(\Pi_X \rho_X^{\otimes n} \Pi_X) \right]^2 \\ &\le 2^{n[H(X)_\rho + \epsilon]} \left\{ 2^{-n[H(X)_\rho - \epsilon]} \right\}^2 \\ &\le 2^{-n[H(X)_\rho - 3\epsilon]} \end{aligned}$$

MORAL: IF $X_{typ} = \mathrm{range}\, \Pi_X$ AND $\rho_{typ} = \Pi_X \rho_X^{\otimes n} \Pi_X$ THEN

$\dim X_{typ} \sim 2^{n[H(X)_\rho + \epsilon]}$

$\mathrm{tr}\, \rho_{typ}^2 \sim 2^{-n[H(X)_\rho - 3\epsilon]}$

APPLY TO DECOUPLING TH'M

SPS EVERYTHING IS TYPICAL:

$$|\psi\rangle_{RAB} \propto (\Pi_R \otimes \Pi_A \otimes \Pi_B) |\varphi\rangle_{RAB}^{\otimes n}$$

THEN:

$$\begin{aligned} \log\left\{ \frac{d_R\, d_A}{d_{A_1}^2} \mathrm{tr}\left[(\psi_{RA})^2 \right] \right\} &= \log d_R + \log d_A + \log \mathrm{tr}\left[(\psi_{RA})^2 \right] - 2 \log d_{A_1} \\ &\qquad (|\psi\rangle \text{ IS PURE}) \\ &= \log d_R + \log d_A + \log \mathrm{tr}\left[(\psi_B)^2 \right] - 2 \log d_{A_1} \\ &\sim n H(R)_\varphi + n H(A)_\varphi - n H(B)_\varphi - 2 \log d_{A_1} \\ &= n H(R)_\varphi + n H(A)_\varphi - n H(RA)_\varphi - 2 \log d_{A_1} \\ &= n I(R;A)_\varphi - 2 \log d_{A_1} \end{aligned}$$

SO, GOOD DECOUPLING PROVIDED

$$\boxed{2 \log d_{A_1} \gg n\, I(R;A)_\varphi}$$

EXACTLY RATE REQUIRED FOR STATE TRANSFER!

ASIDE: INTEGRATION ON THE UNITARY GROUP

WARM-UP: INTEGRATION ON A SPHERE

WHAT DO WE MEAN BY UNIFORM?

INVARIANCE UNDER ROTATIONS

$$\int_{S^d} f(\vec{x})\, d\vec{x} = \int_{S^d} f(R\vec{x})\, d\vec{x} \quad \forall \text{ ROTATIONS } R.$$

GROUP INTEGRATION AND SYMMETRY

NATURAL GROUP SYMMETRIES:

- LEFT AND RIGHT MULTIPLICATION
- UNITARY GROUP HAS A UNIQUE MEASURE INVARIANT UNDER BOTH (HAAR MEASURE)

$$\left.\begin{aligned}\int_{\mathcal{U}(d)} f(g)\, dg &= \int_{\mathcal{U}(d)} f(hg)\, dg \\ &= \int_{\mathcal{U}(d)} f(gh)\, dg\end{aligned}\right\} \forall h \in \mathcal{U}(d)$$

NORMALIZATION: FIX $\int_{\mathcal{U}(d)} 1\, dg = 1.$

THESE RELATIONS ARE SUFFICIENT FOR CALCULATION

CHEAT SHEET: $\int f(g)\, dg = \int f(hg)\, dg = \int f(gh)\, dg \quad \forall h$

$\int dg = 1$

EXAMPLE 1: LET $\mathcal{D}(X) = \int u X u^\dagger du.$

THEN $\mathcal{D}(X) = \mathrm{tr}(X)\frac{I}{d}.$ ie $\mathcal{D}$ IS COMPLETELY DEPOLARIZING

TO SEE THIS, NOTE THAT:

- $$\begin{aligned} V\mathcal{D}(X)V^\dagger &= V\left\{\int u X u^\dagger du\right\}V^\dagger \\ &= \int (Vu) X (Vu^\dagger)\, du \quad \text{(BY LINEARITY OF } \textstyle\int du) \\ &= \int u X u^\dagger du \quad \text{(BY INVARIANCE OF } du) \\ &= \mathcal{D}(X) \end{aligned}$$

  SO $\boxed{[V, \mathcal{D}(X)] = 0 \quad \text{FOR ALL UNITARIES } V.}$

  $\Rightarrow \boxed{\mathcal{D}(X) \propto I}$

- BUT $$\begin{aligned} \mathrm{tr}\,\mathcal{D}(X) &= \mathrm{tr}\int u X u^\dagger du \\ &= \int \mathrm{tr}(u X u^\dagger)\, du \quad \text{(BY LINEARITY OF } \textstyle\int du) \\ &= \int \mathrm{tr}(X u^\dagger u)\, du \quad \text{(TRACE CYCLIC)} \\ &= \int \mathrm{tr}\, X\, du \quad (u \text{ UNITARY}) \\ &= \mathrm{tr}\, X \quad \text{(NORMALIZATION } \textstyle\int du = 1) \end{aligned}$$

- CONCLUDE $\mathcal{D}(X) = \mathrm{tr}(X)\frac{I}{d}$

EXAMPLE 2: CALCULATE $\mathcal{I}(X) = \int (U \otimes U) X (U^\dagger \otimes U^\dagger)\, dU$

AS BEFORE, NOTE

$$(V \otimes V)\mathcal{I}(X)(V^\dagger \otimes V^\dagger) = \int (VU \otimes VU) X [(VU)^\dagger \otimes (VU)^\dagger]\, dU$$

$$= \int (U \otimes U) X (U^\dagger \otimes U^\dagger)\, dU \quad \text{(INVARIANCE OF } \textstyle\int dU)$$

$$= \mathcal{I}(X)$$

SO $\boxed{[\mathcal{I}(X), V \otimes V] = 0 \text{ FOR ALL UNITARY } V}$

FIND Y SUCH THAT $[Y, V \otimes V] = 0 \quad \forall V \in \mathcal{U}(d)$

- $Y = I$ WOULD WORK. OTHERS?
- CONSIDER SWAP OPERATOR F DEFINED SO THAT

  $\forall |\varphi\rangle, |\omega\rangle : \quad F|\varphi\rangle|\omega\rangle = |\omega\rangle|\varphi\rangle$

  THEN
  - $(V \otimes V) F |\varphi\rangle|\omega\rangle = (V \otimes V)|\omega\rangle|\varphi\rangle = V|\omega\rangle \otimes V|\varphi\rangle$
  - $F(V \otimes V)|\varphi\rangle|\omega\rangle = F(V|\varphi\rangle \otimes V|\omega\rangle) = V|\omega\rangle \otimes V|\varphi\rangle$

  SO $[F, V \otimes V] = 0 \quad \forall V \in \mathcal{U}(d)$
- TAKE LINEAR COMBINATIONS $Y = \alpha I + \beta F$.

  THAT'S IT! PROOFS:
  - LINEAR ALGEBRA BRUTE FORCE
  - REPRESENTATION THEORY (BETTER)

EXAMPLE 2 (CONT.D.): $\mathcal{I}(X) := \int (U \otimes U) X (U^\dagger \otimes U^\dagger)\, dU = \alpha(X) I + \beta(X) F$

MORE ON F:

NOTE $F^2 = I$ SO $\mathrm{EIG}(F) = \{\pm 1\}$

LET S_+ BE THE $+1$ (SYMMETRIC) EIGENSPACE

LET S_- BE THE -1 (ANTISYMMETRIC) EIGENSPACE

LET Π_+ AND Π_- BE ORTHOGONAL PROJECTORS ONTO S_+ AND S_-

THEN: $I = \Pi_+ + \Pi_-$ AND $F = \Pi_+ - \Pi_-$

SO $\Pi_\pm = \dfrac{I \pm F}{2}$

BACK TO $\mathcal{I}(X)$:

CAN WRITE $\mathcal{I}(X)$ IN TERMS OF $\Pi_\pm$.

SINCE $\int (U \otimes U) \Pi_\pm (U^\dagger \otimes U^\dagger)\, dU = \Pi_\pm$, GET $\boxed{\mathcal{I}^2 = \mathcal{I}}$.

ALSO, $\mathcal{I}$ IS TRACE-PRESERVING.

CONCLUDE: $\boxed{\mathcal{I}(X) = \mathrm{tr}(X\Pi_+)\dfrac{\Pi_+}{\mathrm{rk}\,\Pi_+} + \mathrm{tr}(X\Pi_-)\dfrac{\Pi_-}{\mathrm{rk}\,\Pi_-}}$

FINDING $\mathrm{rk}\,\Pi_\pm$:

NOTE $S_- = \mathrm{span}\{|i\rangle|j\rangle - |j\rangle|i\rangle \,;\, 1 \le i < j \le d\}$

SO $\mathrm{rk}\,\Pi_- = \dim S_- = \binom{d}{2} = \dfrac{d(d-1)}{2}$

$\mathrm{rk}\,\Pi_+ = d - \dim S_- = \dfrac{d(d+1)}{2}$

ANOTHER LITTLE LEMMA: $\mathrm{tr}\, X^2 = \mathrm{tr}\left[(X \otimes X) F\right]$

PROOF: $\mathrm{tr}\, X^2 = \sum_{ij} X_{ij} X_{ji}$

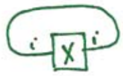
$$\mathrm{tr}\left[(X\otimes X)F\right] = \mathrm{tr}\left[\left(\sum_{ij} X_{ij} |i\rangle\langle j| \otimes \sum_{kl} X_{kl} |k\rangle\langle l|\right) F\right]$$

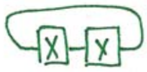
$$= \sum_{ijkl} X_{ij} X_{kl}\, \mathrm{tr}\, |ik\rangle\langle lj| \quad \text{(DEFN OF F)}$$

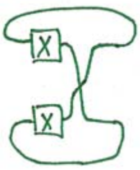
$$= \sum_{ijkl} X_{ij} X_{kl}\, \delta_{il}\, \delta_{kj}$$

$$= \sum_{ij} X_{ij} X_{ji}$$

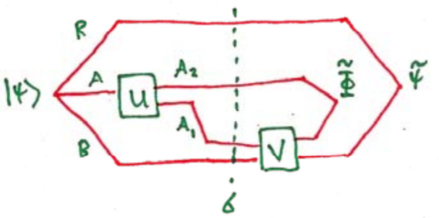
GRAPHICAL PROOF: $\overset{i}{-}\boxed{X}\overset{j}{-} = X_{ij}$

$\mathrm{tr}\, X =$ 

$\mathrm{tr}\, X^2 =$ 

$\mathrm{tr}\left[(X\otimes X)F\right] =$ 

DECOUPLING THEOREM: $\int \left\| \sigma(U)_{RA_2} - \psi_R \otimes I_{A_2}/d_{A_2} \right\|_1^2 dU \le \frac{d_R d_A}{d_{A_1}^2} \mathrm{tr}\left[(\psi_{RA})^2\right]$



PROOF: WILL WORK WITH $\|X\|_2 = \sqrt{\mathrm{tr}\, X^\dagger X}$ INSTEAD OF $\|X\|_1 = \mathrm{tr}\sqrt{X^\dagger X}$

NOTE $\left\| \sigma(U)_{RA_2} - \psi_R \otimes I_{A_2}/d_{A_2} \right\|_2^2$

$$= \mathrm{tr}\left[(\sigma(U)_{RA_2})^2\right] - 2\, \mathrm{tr}\left[(\sigma(U)_{RA_2})(\psi_R \otimes I_{A_2}/d_{A_2})\right] + \mathrm{tr}\left[(\psi_R \otimes I_{A_2}/d_{A_2})^2\right]$$

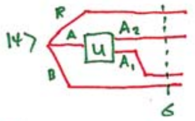
?

$$= \mathrm{tr}\left[(\sigma(U)_{RA_2})^2\right] - \frac{2}{d_{A_2}} \mathrm{tr}\left[(\psi_R)^2\right] + \frac{1}{d_{A_2}^2} \mathrm{tr}\left[(\psi_R)^2\right] \cdot \mathrm{tr}\left[(I_{A_2})^2\right]$$

$$= \mathrm{tr}\left[(\sigma(U)_{RA_2})^2\right] - \frac{1}{d_{A_2}} \mathrm{tr}\left[(\psi_R)^2\right]$$

WHY ? : USES $\mathrm{tr}\left[X_{12}(Y_1 \otimes I_2)\right] = \mathrm{tr}\, X_1 Y_1$ AND $\sigma_R = \psi_R$.

MUST EVALUATE $\int \mathrm{tr}\left[(\sigma(U)_{RA_2})^2\right] dU$



$$\int \mathrm{tr}\left[(\sigma_{RA_2})^2 \right] dU$$

$$= \int \mathrm{tr}\left[(\sigma_{RA_2} \otimes \sigma_{R'A_2'})(F_{RR'} \otimes F_{A_2A_2'}) \right] dU \quad \text{(BY LITTLE LEMMA)}$$

$$= \int \mathrm{tr}\left[(\sigma_{RA} \otimes \sigma_{R'A'})(F_{RR'} \otimes F_{A_2A_2'} \otimes I_{A_1A_1'}) \right] dU \quad \text{(DEFN PARTIAL TRACE)}$$

$$= \int \mathrm{tr}\left[(U_A \psi U_A^\dagger \otimes U_{A'} \psi U_{A'}^\dagger)(F_{RR'} \otimes F_{A_2A_2'} \otimes I_{A_1A_1'}) \right] dU \quad \text{(DEFN OF } \sigma\text{)}$$

$$= \int \mathrm{tr}\left[(\psi_{RA} \otimes \psi_{R'A'})(U_A^\dagger \otimes U_{A'}^\dagger)(F_{RR'} \otimes F_{A_2A_2'} \otimes I_{A_1A_1'})(U_A \otimes U_{A'}) \right] dU \quad \text{(TRACE CYCLIC)}$$

$$= \mathrm{tr}\left\{ (\psi_{RA} \otimes \psi_{R'A'}) \left[\int (U_A^\dagger \otimes U_{A'}^\dagger)(I_{A_1A_1'} \otimes F_{A_2A_2'})(U_A \otimes U_{A'})\, dU \otimes F_{RR'} \right] \right\} \quad \text{(LINEARITY OF } \textstyle\int dU\text{)}$$

$$= \mathrm{tr}\left\{ (\psi_{RA} \otimes \psi_{R'A'}) \left[(\alpha I_{AA'} + \beta F_{AA'}) \otimes F_{RR'} \right] \right\} \quad \text{(WE KNOW THIS INTEGRAL!)}$$

$$= \alpha\, \mathrm{tr}\left[(\psi^R)^2 \right] + \beta\, \mathrm{tr}\left[(\psi_{RA})^2 \right] \quad \text{(BY LITTLE LEMMA AGAIN)}$$

CAN WORK OUT α AND β WITH A BIT MORE WORK.

GET $\beta = \dfrac{d_A d_{A_2} - d_{A_1}}{d_A^2 - 1} \le d_{A_1}$

SIM'LY $\alpha \le \dfrac{1}{d_{A_2}}$

SO: $\displaystyle\int \mathrm{tr}\left[(\sigma_{RA_2})^2 \right] dU \le \frac{1}{d_{A_1}} \mathrm{tr}\left[(\psi_{RA})^2 \right] + \frac{1}{d_{A_2}} \mathrm{tr}\left[(\psi_R)^2 \right]$

ALMOST DONE!

HAVE $\left\| \sigma(U)_{RA_2} - \psi_R \otimes I_{A_2}/d_{A_2} \right\|_2^2$

$$= \mathrm{tr}\left[(\sigma(U))^2_{RA_2} \right] - \frac{1}{d_{A_2}} \mathrm{tr}\left[(\psi_R)^2 \right]$$

AND $\displaystyle\int \mathrm{tr}\left[(\sigma(U)_{RA_2})^2 \right] dU \le \frac{1}{d_{A_1}} \mathrm{tr}\left[(\psi_{RA})^2 \right] + \frac{1}{d_{A_2}} \mathrm{tr}\left[(\psi_R)^2 \right]$

CANCELLATION!

$$\Rightarrow \int \left\| \sigma(U)_{RA_2} - \psi_R \otimes I_{A_2}/d_{A_2} \right\|_2^2 dU$$

$$\le \frac{1}{d_{A_1}} \mathrm{tr}\left[(\psi_{RA})^2 \right] + \frac{1}{d_{A_2}} \mathrm{tr}\left[(\psi_R)^2 \right] - \frac{1}{d_{A_2}} \mathrm{tr}\left[(\psi_R)^2 \right]$$

$$= \frac{1}{d_{A_1}} \mathrm{tr}\left[(\psi_{RA})^2 \right]$$

BUT BY CAUCHY-SCHWARZ, $\displaystyle\sum_{i=1}^m |x_i| = \sum_{i=1}^m |x_i \cdot 1| \le \sqrt{\left(\sum_{i=1}^m x_i^2\right)\left(\sum_{i=1}^m 1\right)} = \sqrt{m}\, \|x\|_2$

SO, $\displaystyle\int \left\| \sigma(U)_{RA_2} - \psi_R \otimes I_{A_2}/d_{A_2} \right\|_1^2 dU$

$$\le d_R d_{A_2} \int \left\| \sigma(U)_{RA_2} - \psi_R \otimes I_{A_2}/d_{A_2} \right\|_2^2 dU \quad \text{(CAUCHY-SCHWARZ)}$$

$$\le d_R d_{A_2} \cdot \frac{1}{d_{A_1}} \mathrm{tr}\left[(\psi_{RA})^2 \right] \quad \text{(ALL OUR HARD WORK)}$$

$$= \frac{d_R d_A}{d_{A_1}^2} \mathrm{tr}\left[(\psi_{RA})^2 \right] \quad \text{(SINCE } d_A = d_{A_1} d_{A_2}\text{)}$$

$\blacksquare$

Further reading on arXiv (beyond Nielsen and Chuang)

- State transfer:
  - Abeyesinghe, Devetak, Hayden and Winter: **The mother of all protocols**
- Resource inequalities
  - Devetak, Harrow and Winter: **A family of quantum protocols**
  - Devetak, Harrow and Winter: **A resource framework for quantum Shannon theory**
- State merging: interpretation of H(A|B)
  - Horodecki, Oppenheim and Winter: **Quantum information can be negative**
- Further development of decoupling approach
  - Oppenheim: **State redistribution as merging**
  - Dupuis: **The decoupling approach to quantum information theory**
- State transfer and black holes:
  - Hayden and Preskill: **Black holes as mirrors**