Distinguishability of Random Unitary Channels

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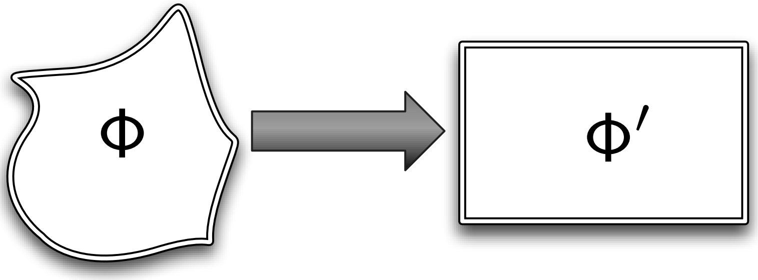
What is this talk about?

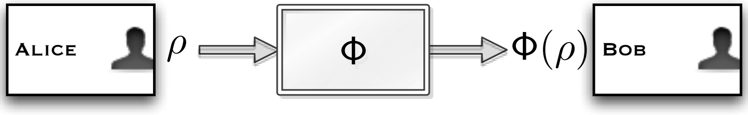


- Given a channel Φ, construct random unitary simulation Φ'
- Simulation is not perfect, but works for several applications

Quantum channels

A channel is a completely positive trace preserving linear map.

- $\operatorname{tr}\Phi(X) = \operatorname{tr} X$
- If $X \geq 0$ then $(\Phi \otimes I_{\mathcal{F}})(X) \geq 0$



- For a channel on states on $\mathcal{A}$, there exists a unitary U on $\mathcal{A} \otimes \mathcal{B}$ with

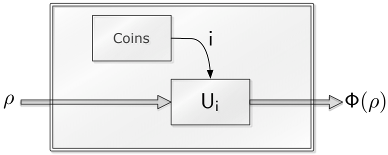
$$\Phi(X) = \operatorname{tr}_{\mathcal{B}} U(X \otimes |0\rangle\langle 0|)U^*.$$

~~Random~~ Mixed-unitary channels

Definition

A channel Φ is *mixed-unitary* if there exists a probability distribution p_i and unitaries U_i such that

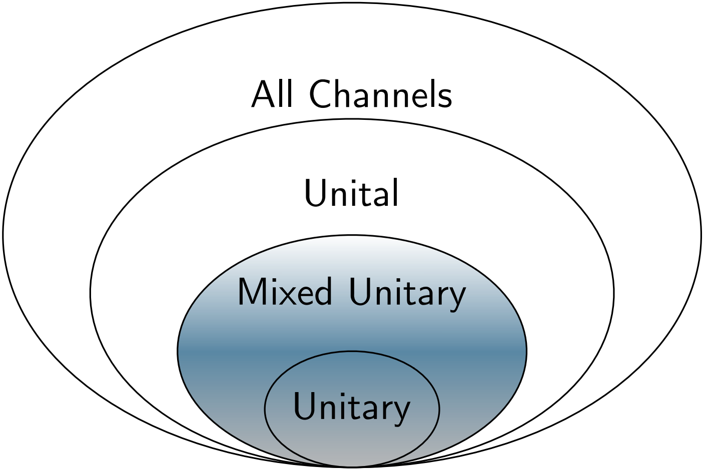
$$\Phi(X) = \sum_i p_i U_i X U_i^*.$$



Examples:

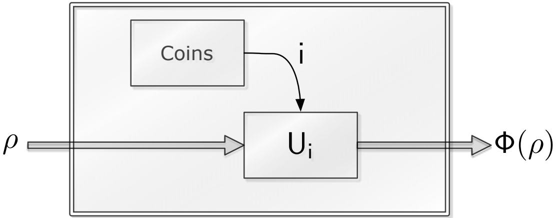
- Depolarizing channel: $N(\rho) = \mathbb{1}/d$
- Dephasing channel: $D(|i\rangle\langle j|) = \delta_{ij}|i\rangle\langle j|$

Classes of channels



- A channel Φ is *unital* if $\Phi(\mathbb{1}) = \mathbb{1}$
- All these containments are strict

Why you should care about mixed-unitary channels



- Exactly reversible using information measured from the environment[1]
- Non-contractive with respect to entropy
- Classical capacity is additive for qubit mixed unitary channels[2]

[1] Gregoratti and Werner 2003
[2] Tregub 1986, King 2002

Measures

- For a channel Φ, how pure can the output be?

$$S_{\min}(\Phi) = \min_{\rho} S(\Phi(\rho))$$

$$\begin{aligned}\|\Phi\|_p &= \max_{\rho} \|\Phi(\rho)\|_p \\ &= \max_{\rho} \left(\operatorname{tr} |\Phi(\rho)|^p\right)^{1/p}\end{aligned}$$

- Given a black box implementing one of two known channels, what is the probability of identifying the black box with one use?

$$\|\Phi - \Psi\|_{\diamond} = \max_{\rho} \|(\Phi \otimes I)(\rho) - (\Psi \otimes I)(\rho)\|_{\mathrm{tr}}$$

Main Result

Theorem

Let $\epsilon > 0$ and Φ, Ψ be channels. Then, for $p < \infty$, there exist mixed-unitary Φ', Ψ' such that

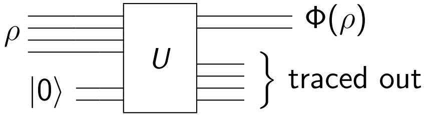
1. $S_{\min}(\Phi) \geq S_{\min}(\Phi') - \log d_\epsilon \geq S_{\min}(\Phi) - \epsilon$
2. $\|\Phi\|_p \leq \frac{\|\Phi'\|_p}{\|\mathbb{1}_{d_\epsilon}/d_\epsilon\|_p} \leq \|\Phi\|_p + \epsilon$
3. $\|\Phi - \Psi\|_\diamond \leq \|\Phi' - \Psi'\|_\diamond \leq \|\Phi - \Psi\|_\diamond + \epsilon$

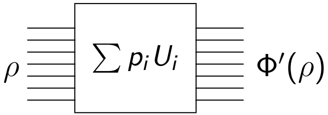
- This generalizes a result of Fukuda on unital channels

Overview

Proof strategy:

- Given a channel Φ, find an approximation Φ' that is mixed-unitary



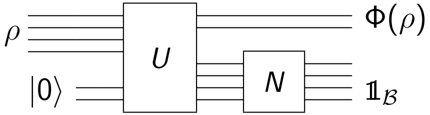


Only two operations that are not mixed-unitary:

1. Partial trace
2. Ancillary qubits in $|0\rangle$ state

Approximating the partial trace

- Replace $\mathrm{tr}_{\mathcal{B}}$ with the completely noisy channel on $\mathcal{B}$



- The resulting output is $\Phi(\rho) \otimes \mathbb{1}_{\mathcal{B}}$
- The depolarizing channel can be written as

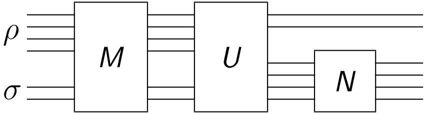
$$N(\rho) = \int U \rho U^* \, \mathrm{d}U \, = \frac{1}{d^2} \sum_{i=1}^{d^2} W_i \rho W_i^* = \mathbb{1}/d$$

for a suitable choice of operators W_i.

Simulating ancillary qubits

To simulate ancillary space:

- Add extra 'input' qubits
- Test that these qubits are in the $|0\rangle$ state
  - If they are, do nothing
  - If not, send all input qubits to highly mixed state

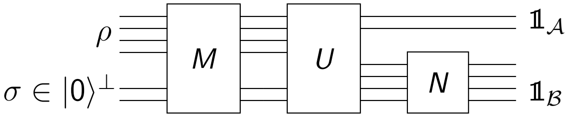


If σ is far from $|0\rangle\langle 0|$, the output has (almost) maximum entropy

- any input maximizing the output norm or minimizing the output entropy will have $\sigma \approx |0\rangle\langle 0|$

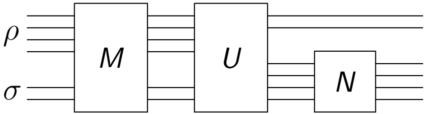
Implementation

The ideal operation M is not unital, and so it is not mixed unitary.



- Mixing operation only needs to increase the entropy, not completely mix the states not of the form $\rho \otimes |0\rangle\langle 0|$
- Solution: completely mix the subspace of states $\mathcal{H} \otimes \{|0\rangle\}^\perp$
- This is the approximation: mixing only this subspace produces a state with trace norm distance $O(1/d)$ to the completely mixed state

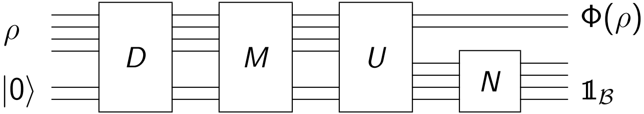
One final piece ...



- Potential problem: entanglement between the subspaces $\mathcal{H} \otimes \{|0\rangle\}$ and $\mathcal{H} \otimes \{|0\rangle\}^{\perp}$ complicates the argument
- Solution: apply dephasing between these subspaces
- Implementation: apply phase flip to $\mathcal{H} \otimes \{|0\rangle\}^{\perp}$ with probability $1/2$

The final construction

- Given input $\rho \otimes |0\rangle\langle 0|$ the output is $\Phi(\rho) \otimes \mathbb{1}_{\mathcal{B}}$
- If the input is not in $\mathcal{H} \otimes \{|0\rangle\}$, the output is highly mixed



- The result is random unitary, since all of the components are.

Main Result

Theorem

Let Φ, Ψ be channels with input plus ancillary dimension d, and let Φ', Ψ' be mixed unitary approximations. Then

1. $S_{\min}(\Phi) \geq S_{\min}(\Phi') - \log d \geq S_{\min}(\Phi) - O(\log d/d)$
2. $\|\Phi\|_p \leq \frac{\|\Phi'\|_p}{\|\mathbb{1}_d/d\|_p} \leq \|\Phi\|_p + O(d^{-1/p})$
3. $\|\Phi - \Psi\|_\diamond \leq \|\Phi' - \Psi'\|_\diamond \leq \|\Phi - \Psi\|_\diamond + O(1/d)$

- Adding (unused) ancillary dimension improves the simulation

Applications

- The **QIP**-hard computational problem of distinguishing two circuits Q_1, Q_2 is to decide between
  1. $\| Q_1 - Q_2 \|_\diamond \geq 2 - \epsilon$,
  2. $\| Q_1 - Q_2 \|_\diamond \leq \epsilon$.
- This construction immediately implies the hardness of this problem for circuits Q_1, Q_2 implementing mixed unitary channels.
- This also reduces the additivity of the classical capacity for a channel to the additivity of a set of mixed unitary approximations.
- Further applications?