Post-selection technique with applications to quantum cryptography and the parallel repetition problem

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joint work with

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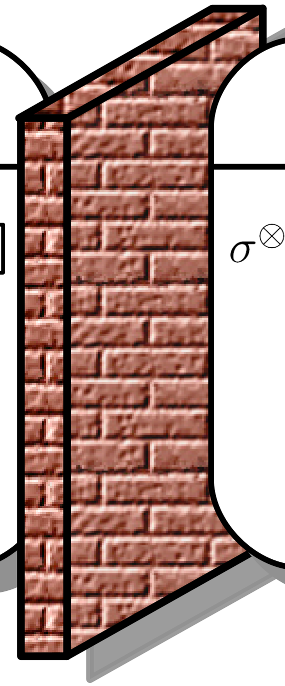
QIP, January 16, 2009, Santa Fe

Goal of Post-Selection Technique

Permutation invariant state

$|\Psi^n\rangle\langle\Psi^n| = \square\square\square\square \cdots \square$

$$\Pr[\mathcal{P}(|\Psi^n\rangle\langle\Psi^n|) = true] \leq \;\; ??$$



Product state

$\sigma^{\otimes n} = \square \otimes \square \otimes \square \otimes \cdots \otimes \square$

$$\Pr[\mathcal{P}(\sigma^{\otimes n}) = true] \leq 2^{-c \cdot n}$$

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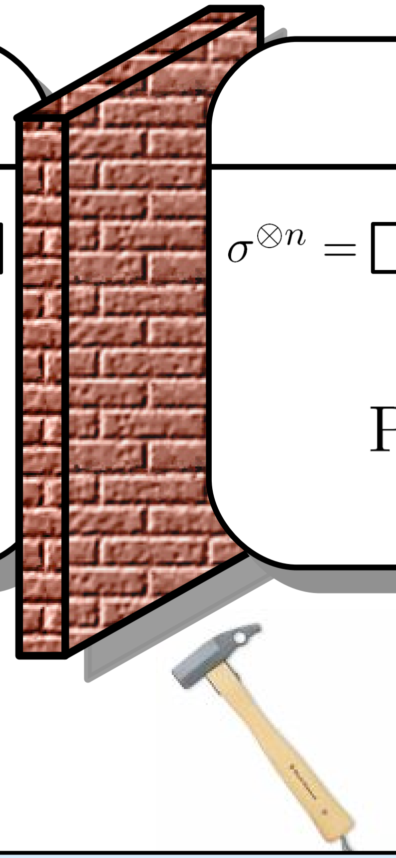

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Post-Selection “Hammer”

Main Result

Permutation invariant state

$|\Psi^n\rangle\langle\Psi^n| = \square\square\square\square\cdots\square$

$$\Pr[\mathcal{P}(|\Psi^n\rangle\langle\Psi^n|) = true] \leq \;??$$

Product state

$\sigma^{\otimes n} = \square \otimes \square \otimes \square \otimes \cdots \otimes \square$

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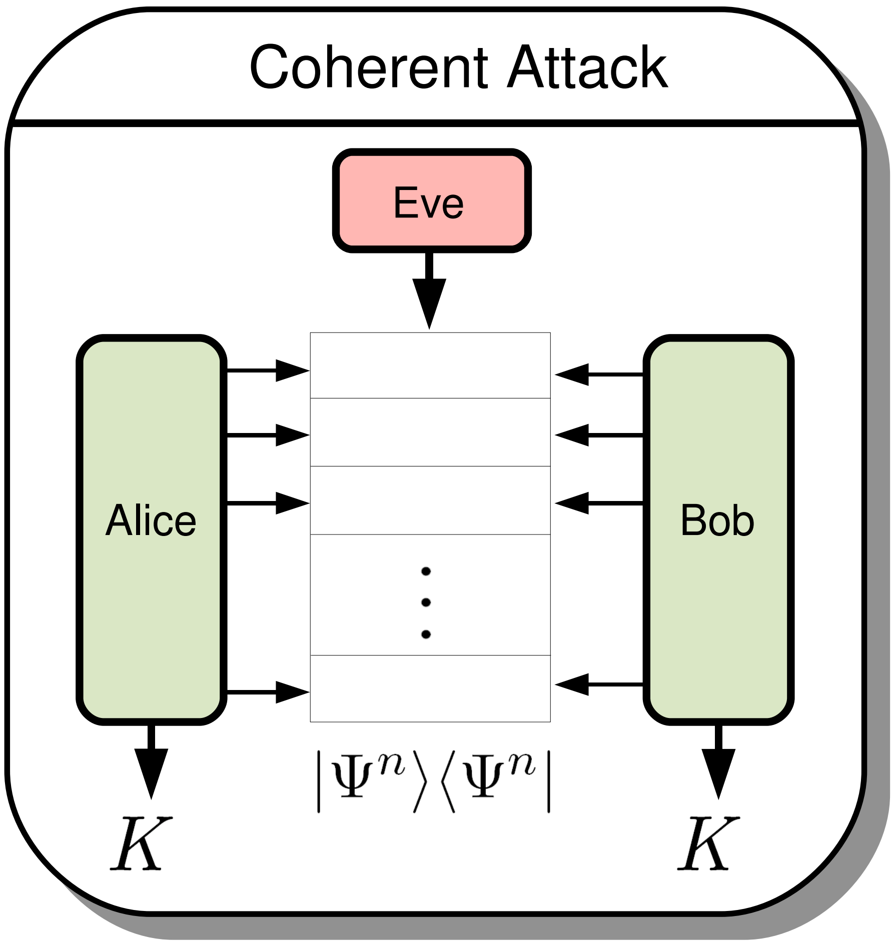
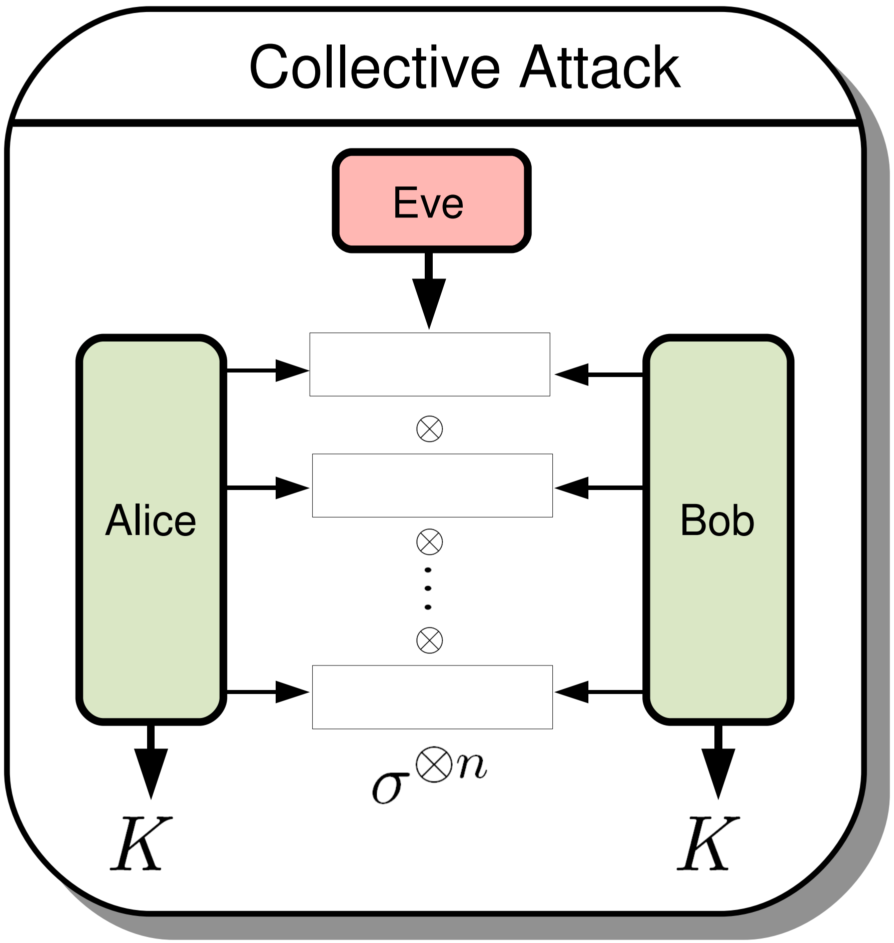
Post-Selection Technique

$$\Pr[\mathcal{P}(|\Psi^n\rangle\langle\Psi^n|) = true] \leq poly(n) \cdot \Pr[\mathcal{P}(\sigma^{\otimes n}) = true]$$

Example 1: Quantum Key Distribution

$\mathcal{P}(|\Psi^n\rangle\langle\Psi^n|) = true \Leftrightarrow$ key generated starting from $|\Psi^n\rangle\langle\Psi^n|$ not secure





$\Pr[\mathcal{P}(|\Psi^n\rangle\langle\Psi^n|) = true] \leq \;??$

$\Pr[\mathcal{P}(\sigma^{\otimes n}) = true] \leq 2^{-c \cdot n}$

(Devetak & Winter, 2005)

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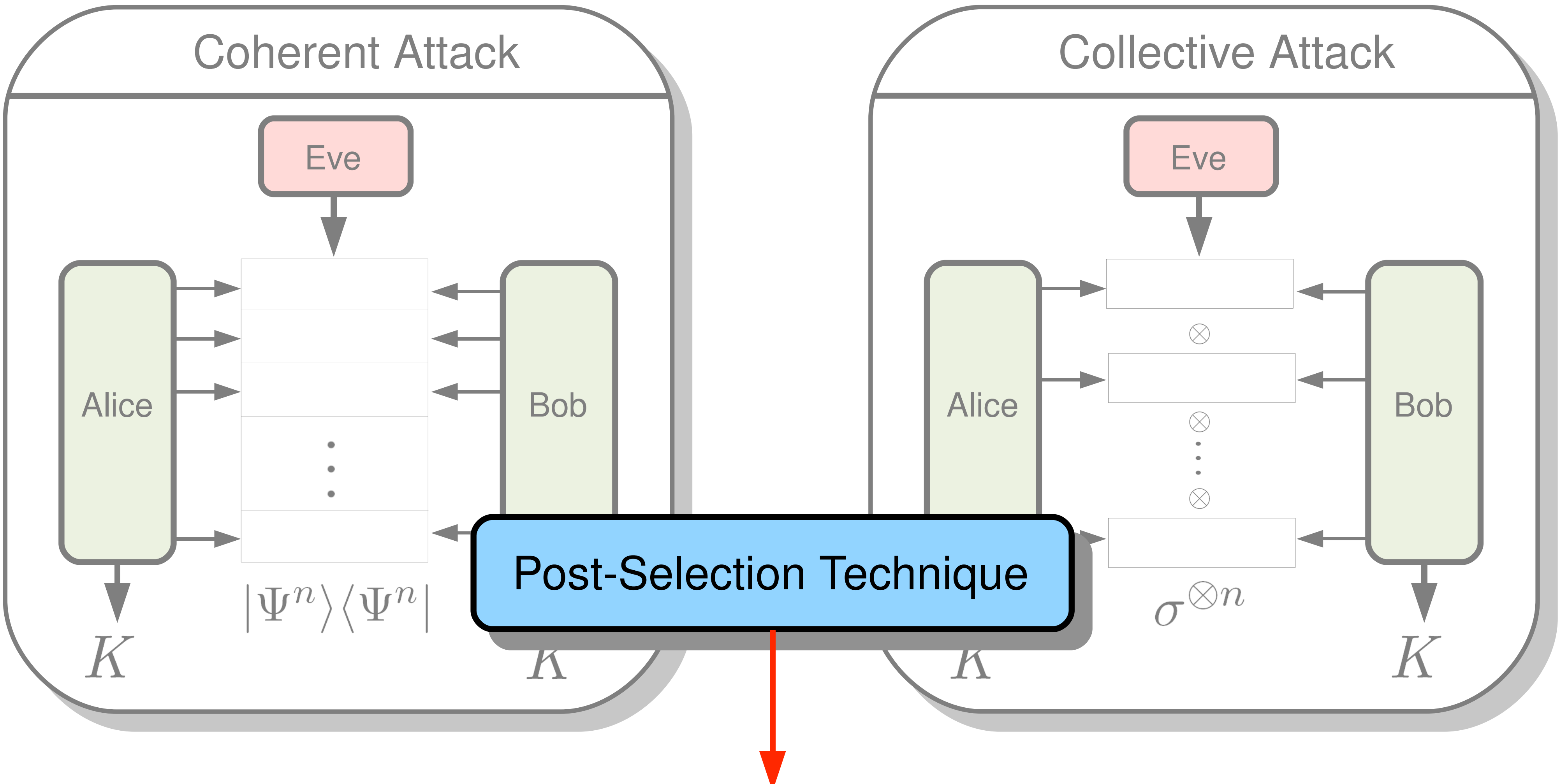
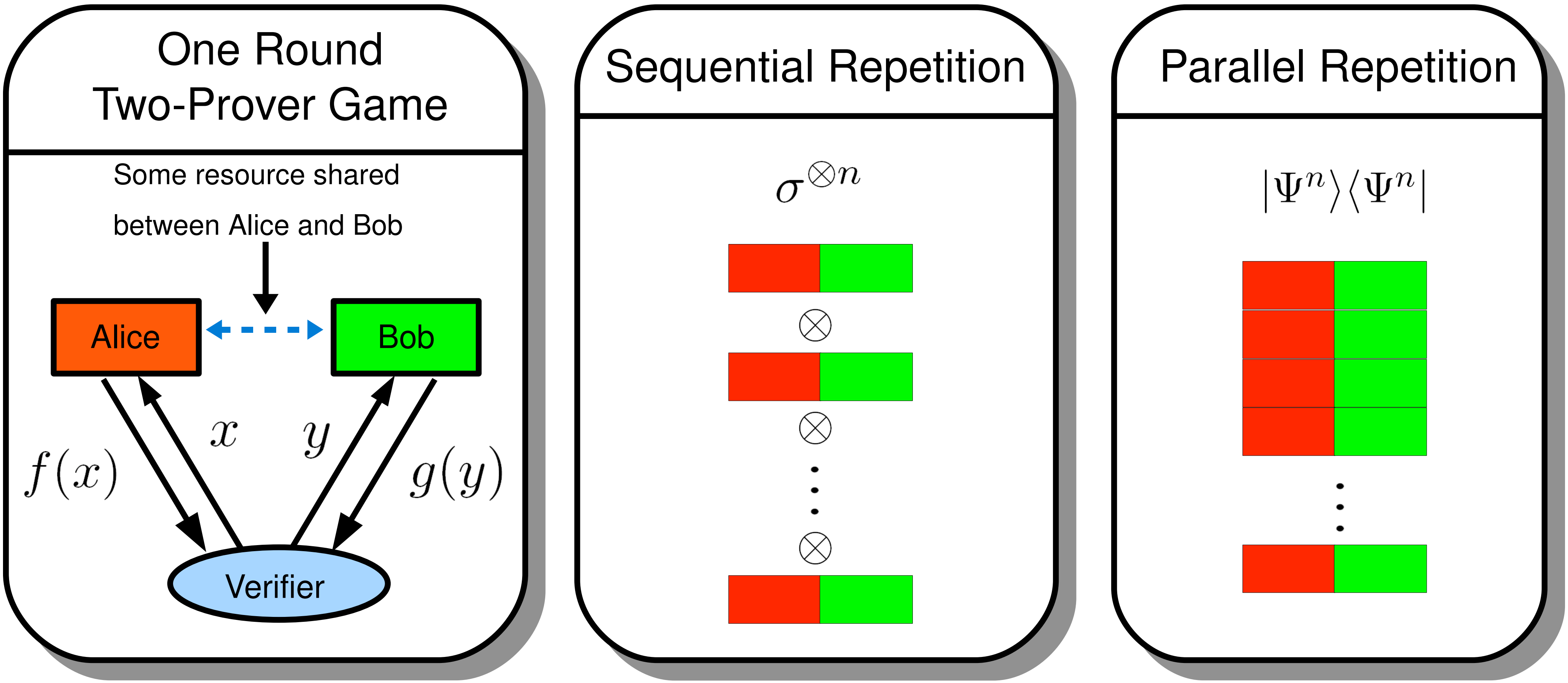


$\Pr[\mathcal{P}(|\Psi^n\rangle\langle\Psi^n|) = true] \leq poly(n) \cdot 2^{-c \cdot n}$ ← $\Pr[\mathcal{P}(\sigma^{\otimes n}) = true] \leq 2^{-c \cdot n}$

Example 2: Parallel Repetition of Two-Prover Games

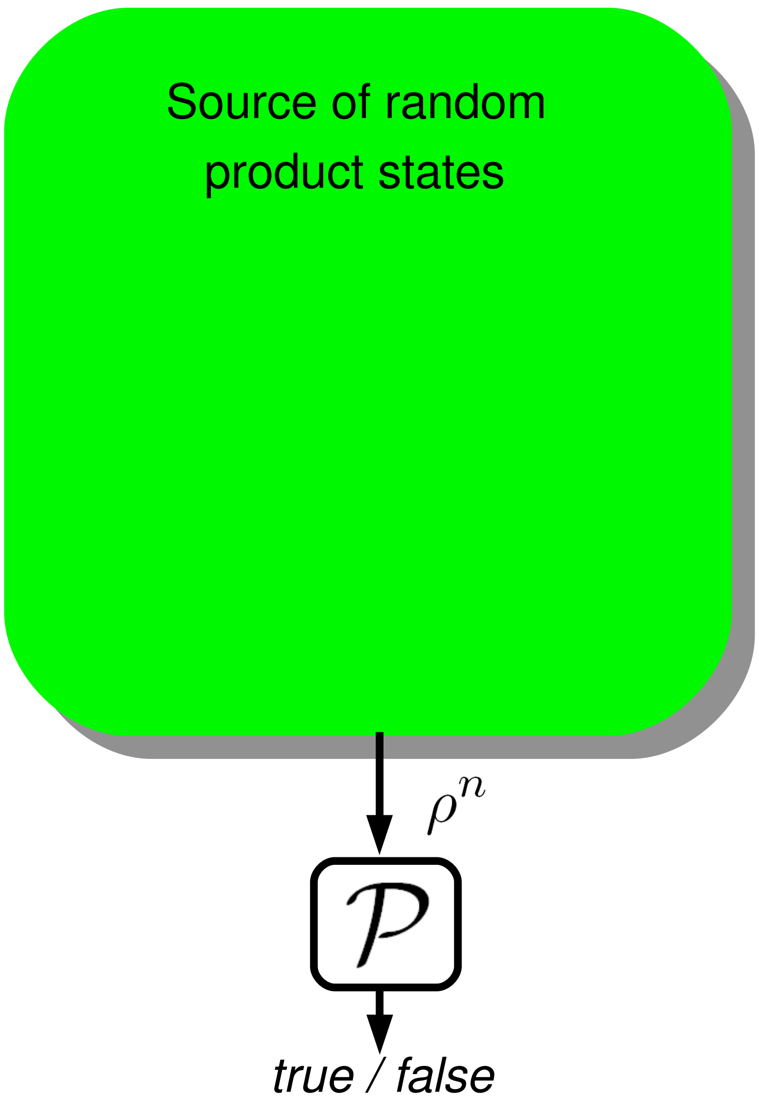


$\Rightarrow$ Post-selection technique can reduce problem to optimization problem over convex set with linear constraint function.

(Raz, 1998 / Holenstein, 2007)

How the Post-Selection Technique Works

To prove: $\Pr[\mathcal{P}(|\Psi^n\rangle\langle\Psi^n|) = true] \leq poly(n) \cdot \Pr[\mathcal{P}(\sigma^{\otimes n}) = true]$



Output of **green** box: $\rho^n := \int \sigma^{\otimes n} \mu(\sigma)$

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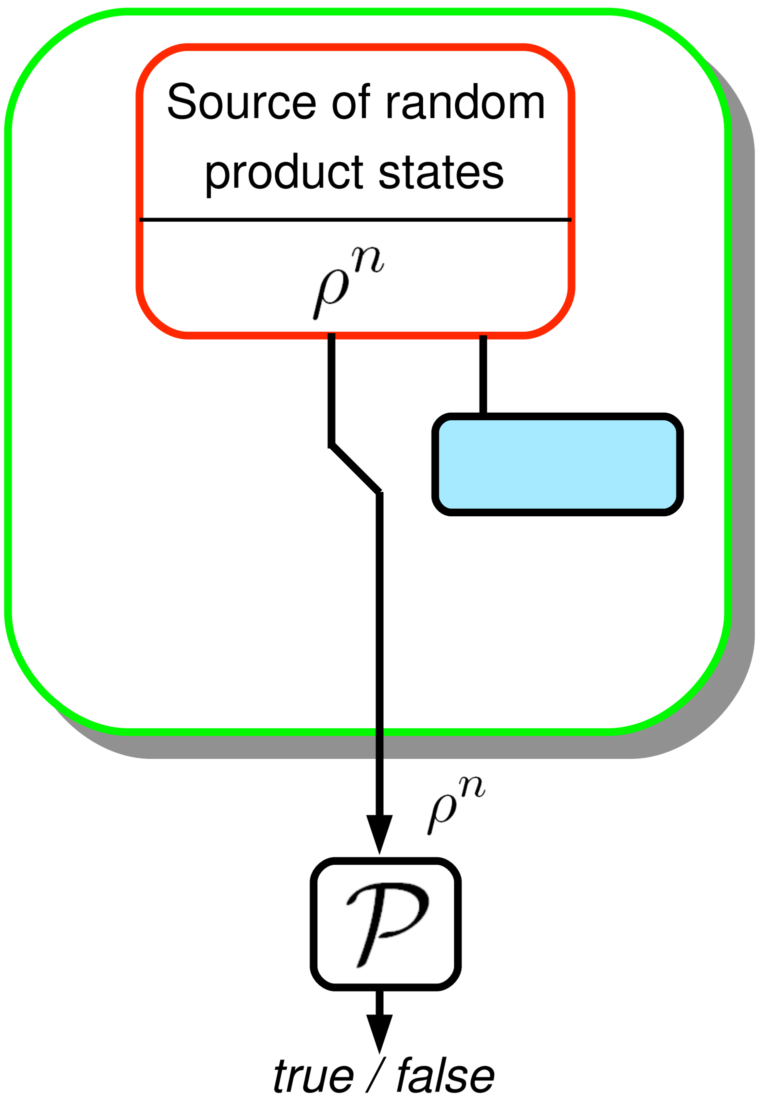

Output of **green** + **red** box: $\rho^n := \int \sigma^{\otimes n} \mu(\sigma)$

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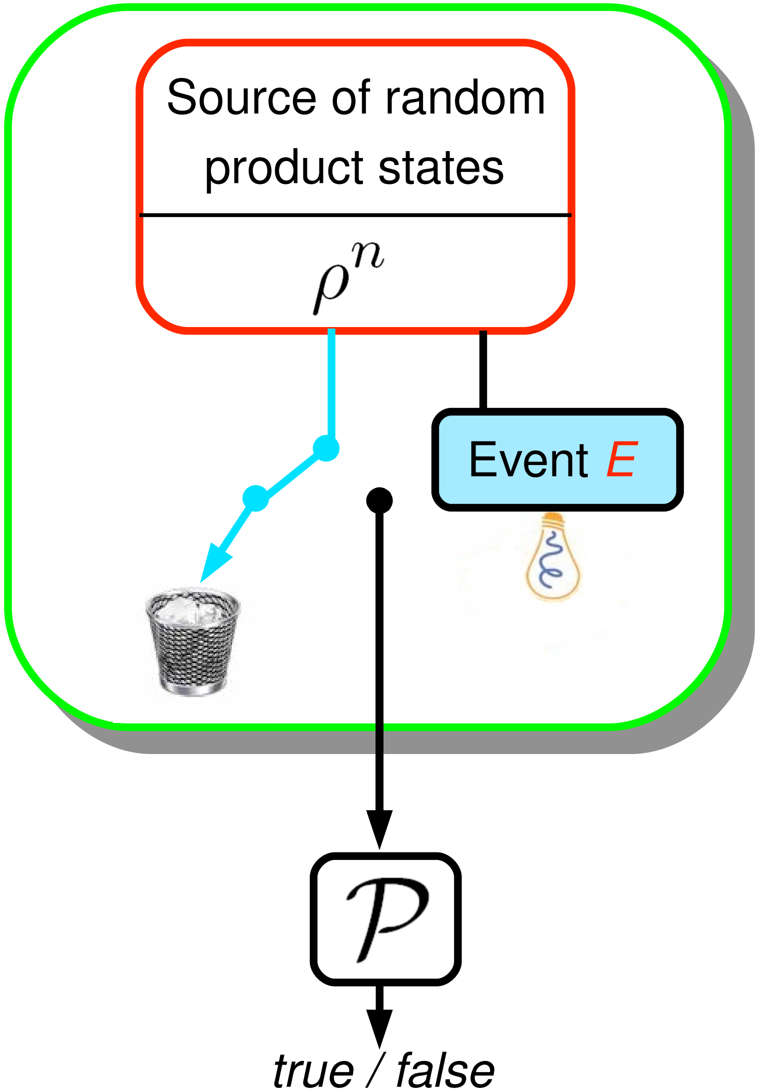


Output of **red** box: $\rho^n := \int \sigma^{\otimes n} \mu(\sigma)$

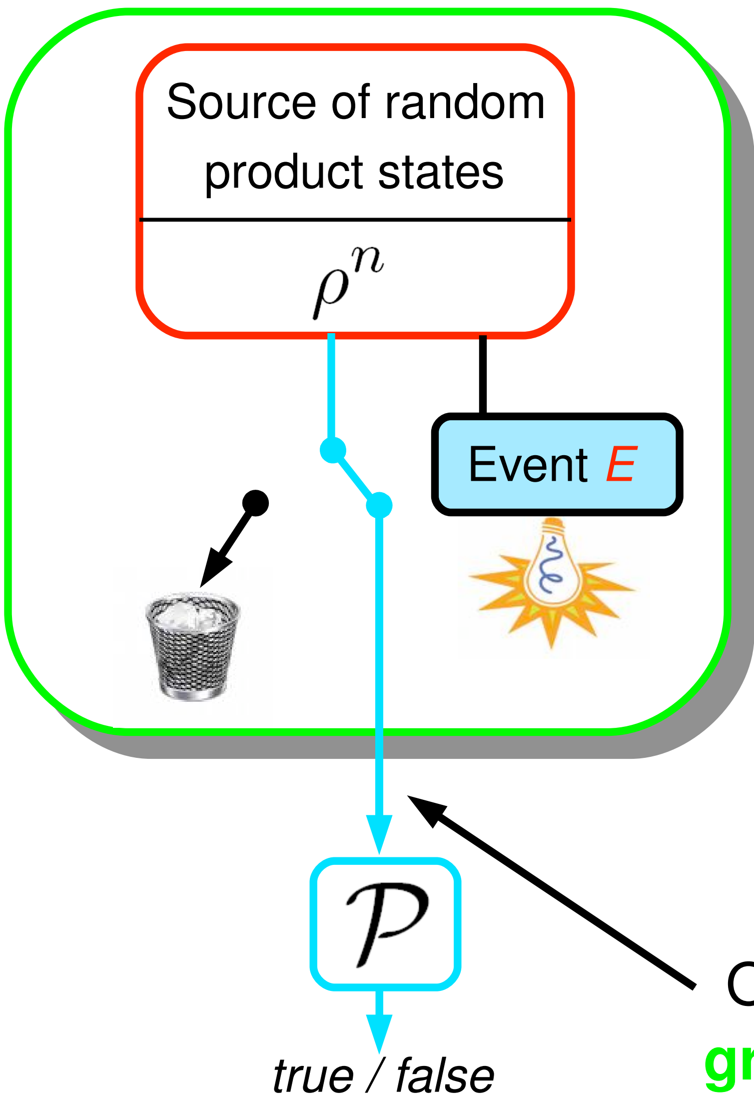
Perform measurement inside the event box:

1. if event *E* does not occur then
   $\Rightarrow$ switch turns to the left

2. if event *E* occurs then
   $\Rightarrow$ switch turns to the right

How the Post-Selection Technique Works

<u>To prove</u>: $\Pr[\mathcal{P}(|\Psi^n\rangle\langle\Psi^n|) = true] \leq poly(n) \cdot \Pr[\mathcal{P}(\sigma^{\otimes n}) = true]$



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<u>Perform measurement inside the event box:</u>

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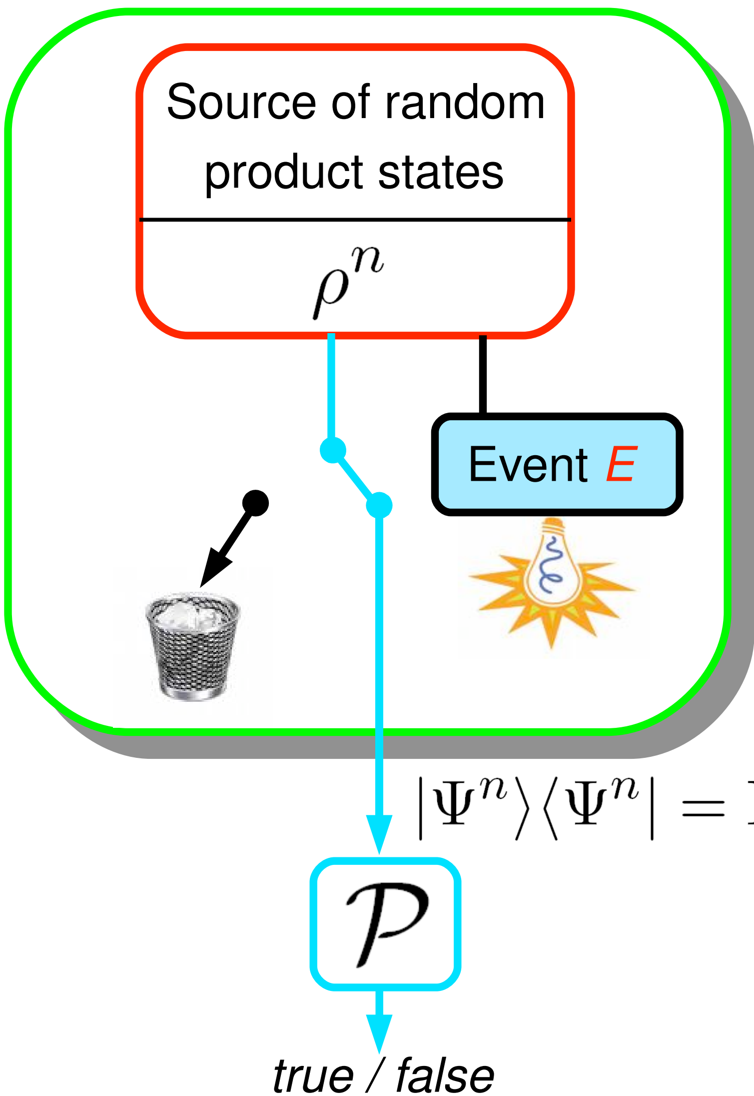
Output of **green** box $= \Pr[E]^{-1} \cdot \mathrm{Tr}_{R^n}\left((id_{\mathcal{H}^n} \otimes E_{R^n})\rho_{\mathcal{H}^n R^n}\right)$

How the Post-Selection Technique Works

<u>To prove</u>: $\Pr[\mathcal{P}(|\Psi^n\rangle\langle\Psi^n|) = true] \leq poly(n) \cdot \Pr[\mathcal{P}(\sigma^{\otimes n}) = true]$

<u>Lemma</u>: There exists a measurement such that if event *E* occurs, the input to $\mathcal{P}$ is $|\Psi^n\rangle$ and $\Pr[E] > 1/poly(n)$



$$|\Psi^n\rangle\langle\Psi^n| = \Pr[E]^{-1} \cdot \mathrm{Tr}_{R^n}\left((id_{\mathcal{H}^n} \otimes E_{R^n})\rho_{\mathcal{H}^n R^n}\right)$$

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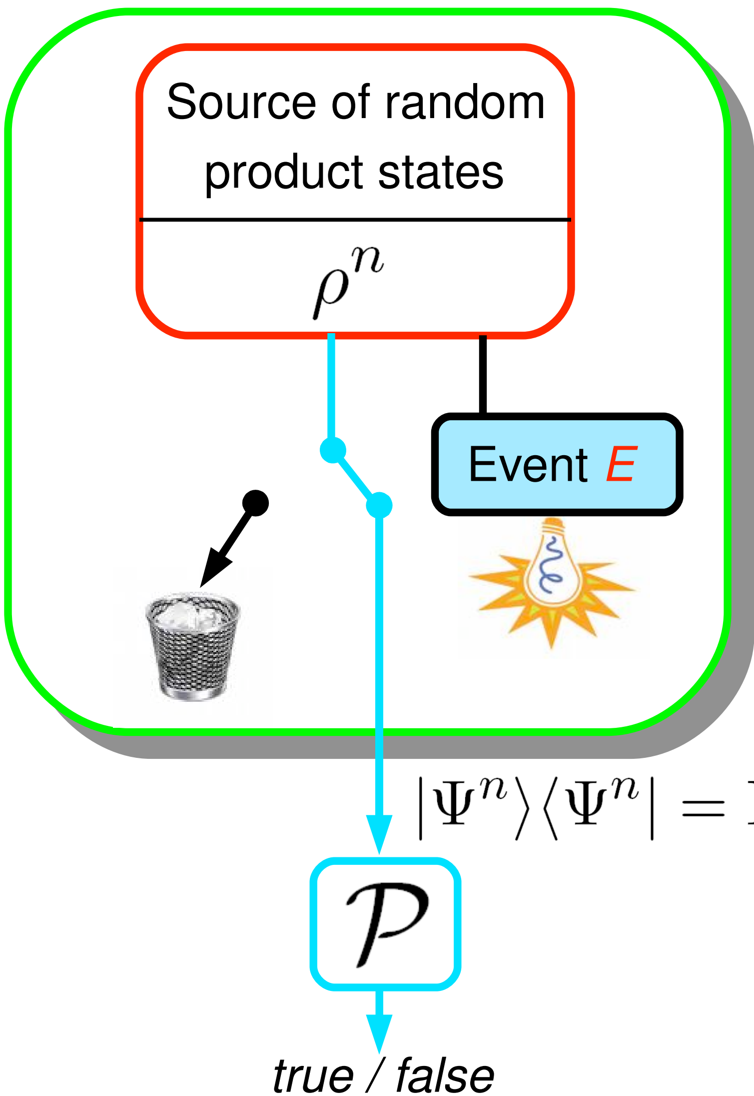

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<u>Remark</u>: The polynomial factor depends on the dimension of the Hilbert space, i.e., $poly(n) \sim n^{\dim(\mathcal{H})}$

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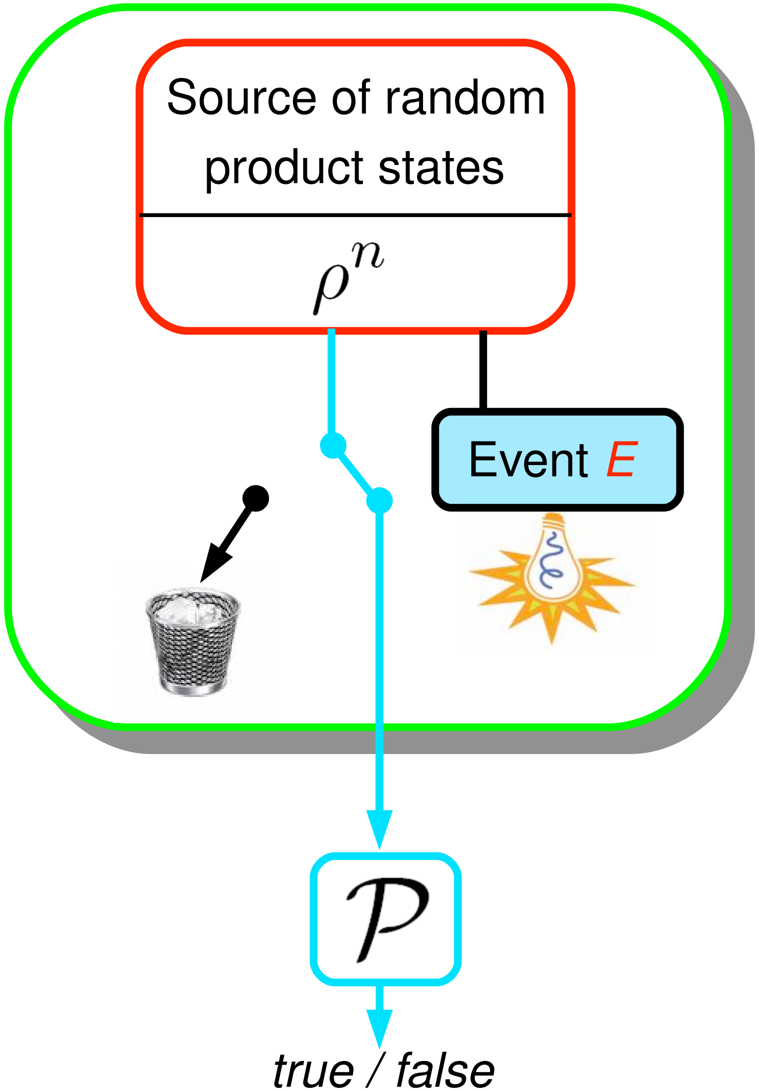

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$$\Pr[\mathcal{P}(|\Psi^n\rangle\langle\Psi^n|) = true] = \Pr[\mathcal{P}(\rho^n) = true|E]$$

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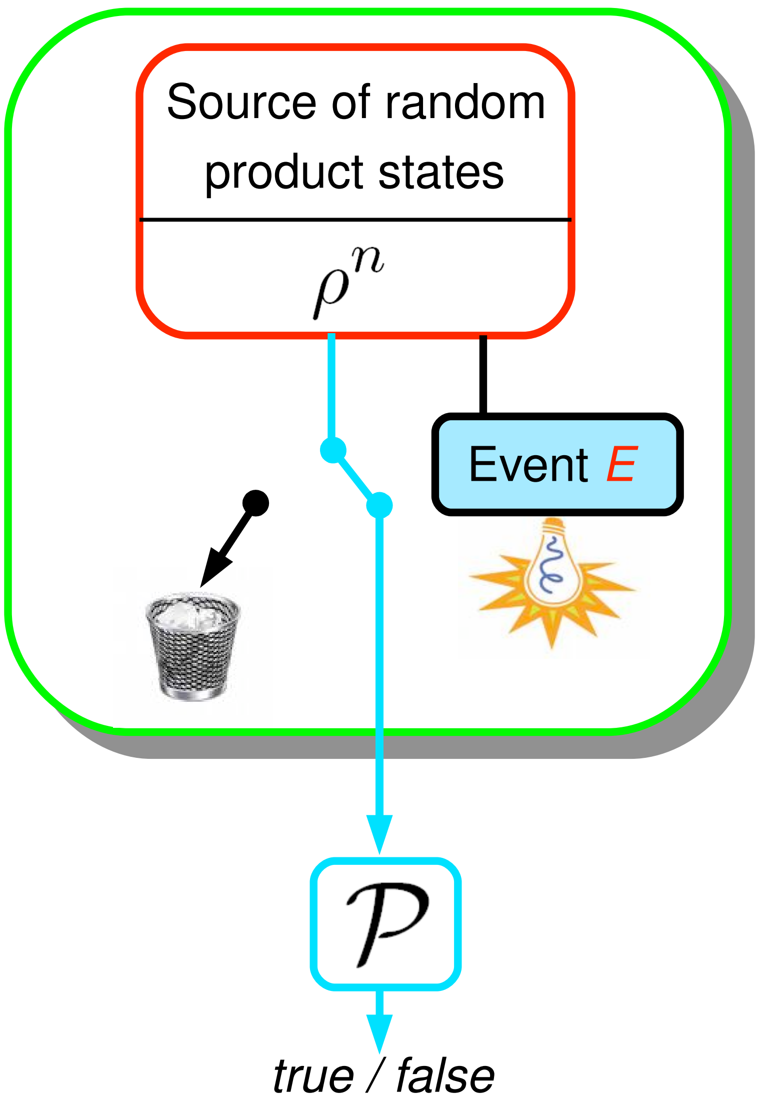

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$$\rho^n := \int \sigma^{\otimes n} \mu(\sigma)$$

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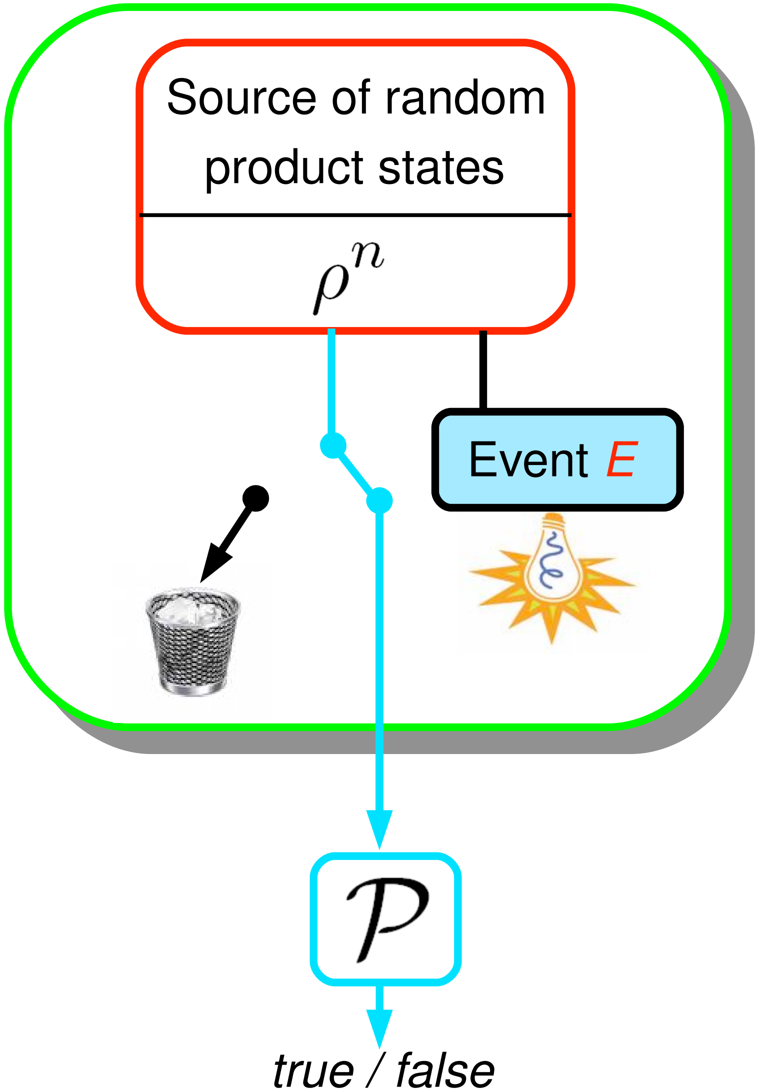

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&\leq \frac{1}{\Pr[E]} \cdot \Pr[\mathcal{P}(\sigma^{\otimes n}) = true]
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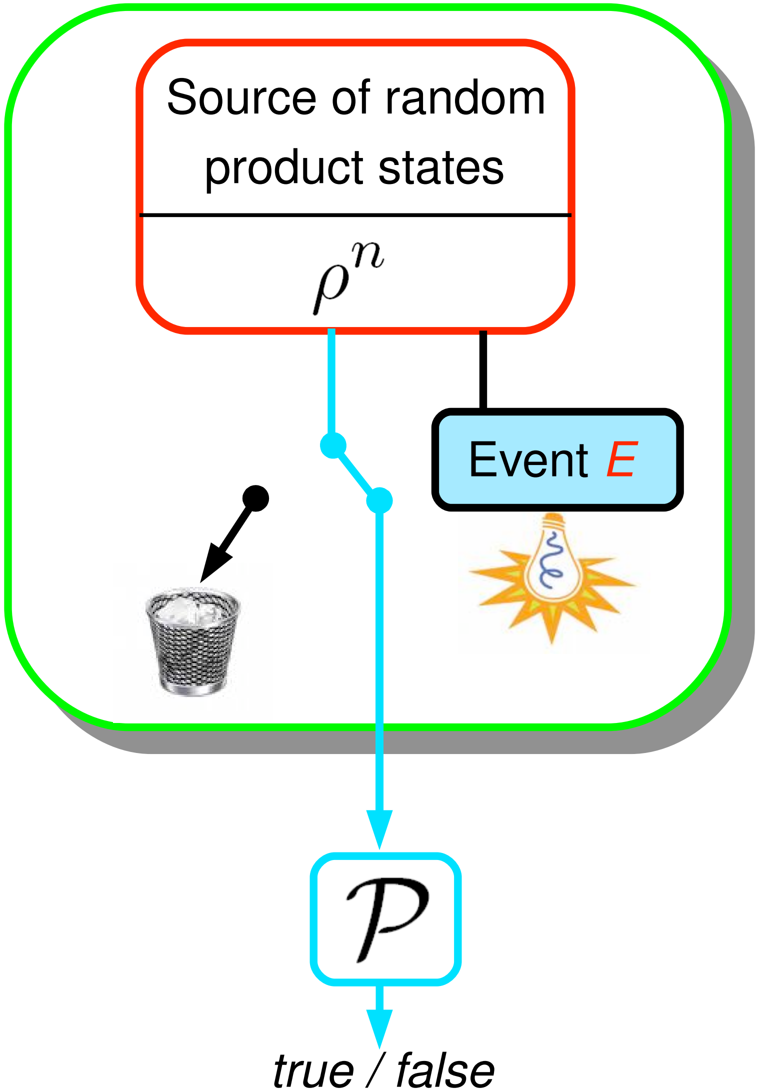

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&\leq poly(n) \cdot \Pr[\mathcal{P}(\sigma^{\otimes n}) = true]
\end{aligned}
$$

Lemma

Conclusions and Open Problems

- Proving upper bound for product states gives upper bound for symmetric states (only polynomially worse)
- Easier to handle than exponential de Finetti theorem
- Gives better bounds

- Simplification for general parallel repetition problems?
- How to generalize the technique to infinite dimensional systems?

Any Questions?

For more information see: arXiv:0809.3019 (Phys. Rev. Lett. 102, 020504 (2009))