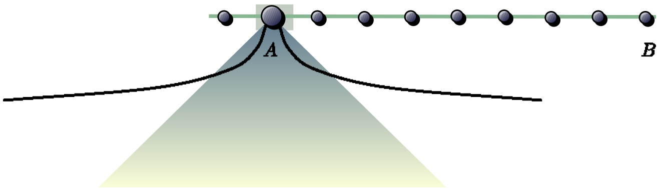
"Supersonic" quantum communication

– arbitrarily fast propagation of signals in lattice models



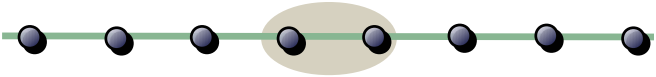
Jens Eisert

David Gross

Outline

- Lieb-Robinson bounds
  - Limit propagation speed of information in spin systems
- Why should I care?
  - What's the connection to quantum information *processing*?
- Supersonic communication
  - Violating Lieb-Robinson bounds

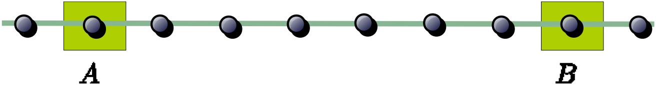
Setting the scene: spin systems on a line



$$\mathbb{1}_{1,\ldots,j-1} \otimes h_j \otimes \mathbb{1}_{j+2,\ldots,n}$$

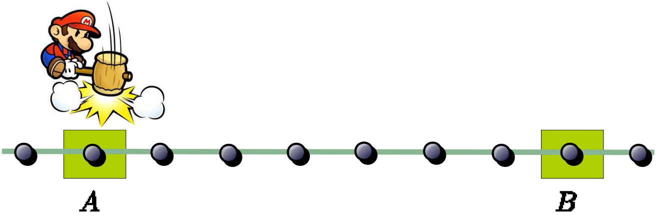
- Hilbert space: $\mathcal{H} = \otimes_{j=1}^{n} \mathbb{C}^d$
- Nearest-neighbor Hamiltonian: $H = \sum_{j=1}^{n-1} h_j$

Propagation of signals



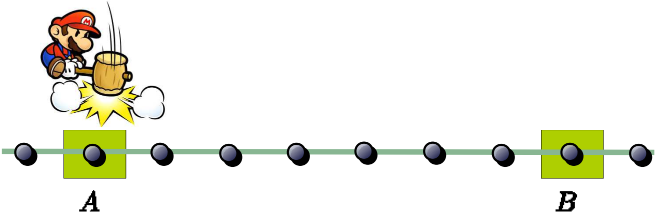
- Initially in ground state
- Apply local unitary operation
- Excitation propagates through chain
- After some time: effect is felt in distant region

Propagation of signals



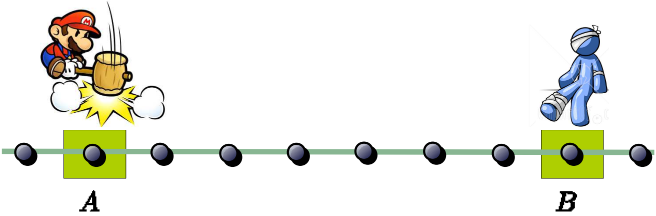
- Initially in ground state
- Apply local unitary operation
- Excitation propagates through chain
- After some time: effect is felt in distant region

Propagation of signals



- Initially in ground state
- Apply local unitary operation
- Excitation propagates through chain
- After some time: effect is felt in distant region

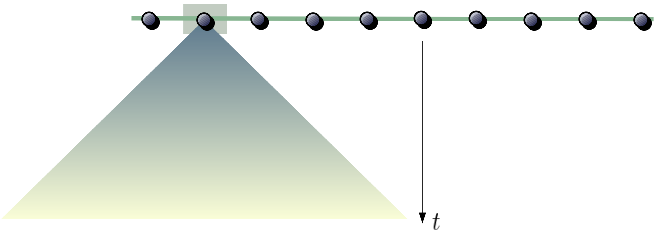
Propagation of signals



- Initially in ground state
- Apply local unitary operation
- Excitation propagates through chain
- After some time: effect is felt in distant region

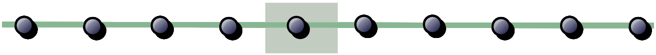
Lieb-Robinson, very informal

- Perturbation propagates at finite “speed of sound” v (up to exponentially small corrections).



- …giving rise to a causal cone.

Lieb-Robinson, slightly less informal

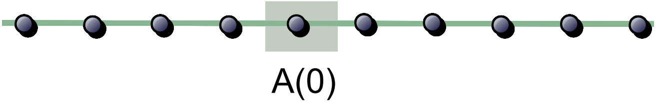


- In Heisenberg-picture, consider localized observable $A(0)$
- Time-evolved observable: $A(t) = e^{itH} A(0) e^{-itH}$

Lieb-Robinson ('72): There is velocity v such that $A(t)$ has support in cone with radius vt, up to exponentially small corrections.

- Speed of sound v depends only on coupling strength $\max_j \|h_j\|$.

Lieb-Robinson, slightly less informal

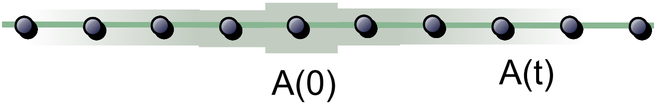


- In Heisenberg-picture, consider localized observable $A(0)$
- Time-evolved observable: $A(t) = e^{itH} A(0) e^{-itH}$

Lieb-Robinson ('72): There is velocity v such that $A(t)$ has support in cone with radius vt, up to exponentially small corrections.

- Speed of sound v depends only on coupling strength $\max_j \|h_j\|$.

Lieb-Robinson, slightly less informal

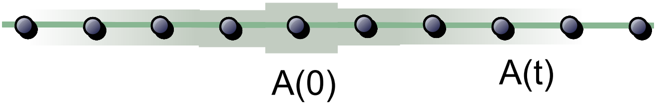


- In Heisenberg-picture, consider localized observable $A(0)$
- Time-evolved observable: $A(t) = e^{itH} A(0) e^{-itH}$

Lieb-Robinson ('72): There is velocity v such that $A(t)$ has support in cone with radius vt, up to exponentially small corrections.

- Speed of sound v depends only on coupling strength $\max_j \|h_j\|$.

Lieb-Robinson, slightly less informal



- In Heisenberg-picture, consider localized observable $A(0)$
- Time-evolved observable: $A(t) = e^{itH} A(0) e^{-itH}$

Lieb-Robinson ('72): There is velocity v such that $A(t)$ has support in cone with radius vt, up to exponentially small corrections.

- Speed of sound v depends only on coupling strength $\max_j \|h_j\|$.

Lieb-Robinson: Applications

Proofs of the following phenomena based on L-R bounds:

- Clustering of correlations in ground states:
  $\langle A \otimes B \rangle - \langle A \rangle \langle B \rangle < e^{-C\,\mathrm{dist}(A,B)}$
- Area laws for the entanglement entropy
- Efficient classical MPS description of ground states for gapped models
- Simulatibility of time evolution on short time scales

C.f. Hastings, Koma, Commm. Math. Phys. (2006); Nachtergaele, Sims, Comm. Math. Phys. (2006); Eisert, Osborne, PRL (2006); Bravyi, Hastings, Verstraete, PRL (2006); Osborne, PRA (2007); Hastings, JSTAT (2007); Schuch, Cirac, Verstraete, PRL (2007); Eisert, Cramer, Plenio, Rev. Mod. Phys. (2009); Gottesman, Hastings, pre-print (2009); Irani, pre-print (2009); ...

Lieb-Robinson: Applications

Proofs of the following phenomena based on L-R bounds:

- Clustering of correlations in ground states;

QIP 09 talks by

- Ashley Montanaro (on Quantum Boolean Functions!),
- Sandy Irani,
- Matt Hastings,
- Norbert Schuch

based on L-R bounds.

Osborne, PRL (2006); Bravyi, Hastings, Verstraete, PRL (2006); Osborne, PRA (2007); Hastings, JSTAT (2007); Schuch, Cirac, Verstraete, PRL (2007); Eisert, Cramer, Plenio, Rev. Mod. Phys. (2009); Gottesman, Hastings, pre-print (2009); Irani, pre-print (2009); . . .

Lieb-Robinson bounds: Validity

Proven instances include

- All finite-dimensional models,

  Lieb, Robinson, Comm. Math. Phys. (1972).

- Infinite-dimensional models with...
  - ...bounded interaction terms (but potentially unbouded on-site terms),

    Nachtergaele, Raz, Schlein, Sims, pre-print (2007).
  - ...unbounded, but harmonic Hamiltonians.

    Nachtergaele, Raz, Schlein, Sims, pre-print (2007); Cramer, Serafini, Eisert, pre-print (2008).

Our agenda: Explore how badly Lieb-Robinson bounds can fail for general, unbounded interactions.

Lieb-Robinson bounds: Validity

Proven instances include

- All finite-dimensional models,

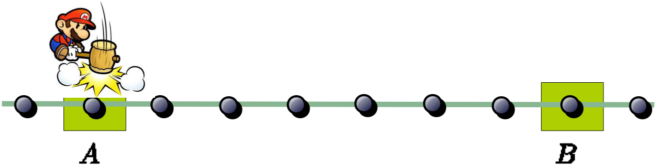
  Lieb, Robinson, Comm. Math. Phys. (1972).
- Infinite-dimensional models with...
  - ...bounded interaction terms (but potentially unbouded on-site terms),

    Nachtergaele, Raz, Schlein, Sims, pre-print (2007).
  - ...unbounded, but harmonic Hamiltonians.

    Nachtergaele, Raz, Schlein, Sims, pre-print (2007); Cramer, Serafini, Eisert, pre-print (2008).

Our agenda: Explore how badly Lieb-Robinson bounds can fail for general, unbounded interactions.

More precise setup



- Set $m = \mathrm{dist}(A, B)$
- Initial state of chain: $|\Psi\rangle$
- At region A: apply a unitary perturbation U_A
- At region B: measure POVM element T_B

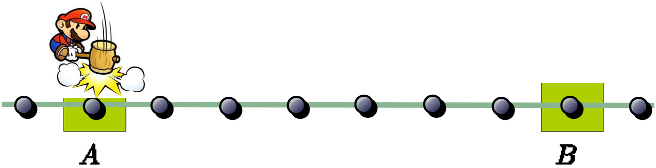
Probabilities of measuring T_B:

$$P_0 = \mathrm{tr}[T_B \, e^{itH} |\Psi\rangle\langle\Psi| e^{-itH}], \qquad P_1 = \mathrm{tr}[T_B \, e^{itH} U_A |\Psi\rangle\langle\Psi| U_A^\dagger e^{-itH}].$$

Relevant parameter:

- Signal strength $\delta = |P_0 - P_1|$.

More precise setup



- Set $m = \mathrm{dist}(A, B)$
- Initial state of chain: $|\Psi\rangle$
- At region A: apply a unitary perturbation U_A
- At region B: measure POVM element T_B

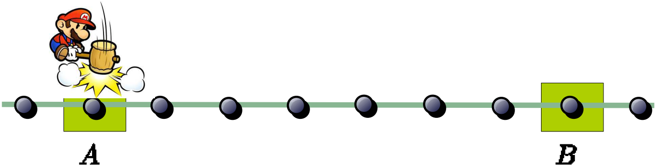
Probabilities of measuring T_B:

$$P_0 = \mathrm{tr}[T_B\, e^{itH} |\Psi\rangle\langle\Psi| e^{-itH}], \qquad P_1 = \mathrm{tr}[T_B\, e^{itH} U_A |\Psi\rangle\langle\Psi| U_A^\dagger e^{-itH}].$$

Relevant parameter:

- Signal strength $\delta = |P_0 - P_1|$.

More precise setup



- Set $m = \text{dist}(A, B)$
- Initial state of chain: $|\Psi\rangle$
- At region A: apply a unitary perturbation U_A
- At region B: measure POVM element T_B

Probabilities of measuring T_B:

$$P_0 = \text{tr}[T_B\, e^{itH}|\Psi\rangle\langle\Psi|e^{-itH}], \qquad P_1 = \text{tr}[T_B\, e^{itH}U_A|\Psi\rangle\langle\Psi|U_A^\dagger e^{-itH}].$$

Relevant parameter:

- Signal strength $\delta = |P_0 - P_1|$.

What would we accept as a fair violation?

- In unbounded models, must put constraints on the energy.
- Should outperform a "tensor product of independent chains"

We prove much more:

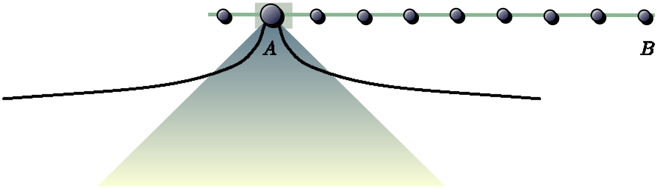
There are models for which the signal strength δ, the involved energies, the perturbation U_A, and the measurement T_B are all constant.
Yet, covered distance m scales *exponentially* in time t.

What would we accept as a fair violation?

- In unbounded models, must put constraints on the energy.
- Should outperform a "tensor product of independent chains"

We prove much more:

> There are models for which the signal strength δ, the involved energies, the perturbation U_A, and the measurement T_B are all constant.
> Yet, covered distance m scales *exponentially* in time t.



...exponentially widened causal cone.

The model: Hamiltonian

- Choose "Fock basis" $\{|0\rangle, |1\rangle, |2\rangle, \dots\}$ and a spin-1 degree of freedom at each site.
- Translationally invariant, nearest-neighbor Hamiltonian

$$H = \sum_{j=1}^{n-1} f_{j,j+1} + \sum_{j=1}^{n} g_j$$

- Where

$$\begin{aligned} f_{j,j+1} &= \sum_{k,l=0}^{\infty} (2l-1)\big(\mathrm{i}A^{\dagger}_{j;l,k} B_{j+1;l,k} + h.c.\big), \\ g_j &= 2\sum_{k=0}^{\infty} (\mathrm{i}k|k+1,\uparrow\rangle\langle k,\downarrow| + h.c.) \\ A_{j;k,l} &= |k,\uparrow\rangle\langle l,\downarrow|, \qquad B_{j;k,l} = |k,\downarrow\rangle\langle l,\downarrow|. \end{aligned}$$

(bear with me :-)).

- Cooked up, but not totally un-natural. Inspired by harmonic hopping terms, after failed attempt to prove L-R for Bose-Hubbard.

The model: Hamiltonian

- Choose "Fock basis" $\{|0\rangle, |1\rangle, |2\rangle, \dots\}$ and a spin-1 degree of freedom at each site.
- Translationally invariant, nearest-neighbor Hamiltonian

$$H = \sum_{j=1}^{n-1} f_{j,j+1} + \sum_{j=1}^{n} g_j$$

- Where

$$\begin{aligned} f_{j,j+1} &= \sum_{k,l=0}^{\infty} (2l-1)\big(\mathrm{i}A^{\dagger}_{j;l,k} B_{j+1;l,k} + h.c.\big), \\ g_j &= 2\sum_{k=0}^{\infty} (\mathrm{i}k|k+1,\uparrow\rangle\langle k,\downarrow| + h.c.) \\ A_{j;k,l} &= |k,\uparrow\rangle\langle l,\downarrow|, \qquad B_{j;k,l} = |k,\downarrow\rangle\langle l,\downarrow|. \end{aligned}$$

(bear with me :-)).

- Cooked up, but not totally un-natural. Inspired by harmonic hopping terms, after failed attempt to prove L-R for Bose-Hubbard.

The model: Hamiltonian

- Choose "Fock basis" $\{|0\rangle, |1\rangle, |2\rangle, \dots\}$ and a spin-1 degree of freedom at each site.
- Translationally invariant, nearest-neighbor Hamiltonian

$$H = \sum_{j=1}^{n-1} f_{j,j+1} + \sum_{j=1}^{n} g_j$$

- Where

$$\begin{aligned} f_{j,j+1} &= \sum_{k,l=0}^{\infty} (2l-1)\big(\mathrm{i}A^{\dagger}_{j;l,k} B_{j+1;l,k} + h.c.\big), \\ g_j &= 2\sum_{k=0}^{\infty} (\mathrm{i}k|k+1,\uparrow\rangle\langle k,\downarrow| + h.c.) \\ A_{j;k,l} &= |k,\uparrow\rangle\langle l,\downarrow|, \qquad B_{j;k,l} = |k,\downarrow\rangle\langle l,\downarrow|. \end{aligned}$$

  (bear with me :-)).
- Cooked up, but not totally un-natural. Inspired by harmonic hopping terms, after failed attempt to prove L-R for Bose-Hubbard.

The model: map to random walk

Initial state and signal state:

$$
\begin{aligned}
|\Psi\rangle &= |0\rangle \otimes |0\rangle \otimes |0\rangle \otimes \ldots, \\
|\Psi_{\mathsf{signal}}\rangle &= |1\rangle \otimes |0\rangle \otimes |0\rangle \otimes \ldots.
\end{aligned}
$$

Signal state couples only to

$$
\begin{aligned}
||1\rangle\rangle &:= |1\rangle \otimes |0\rangle \otimes |0\rangle \otimes \ldots, \\
||2\rangle\rangle &:= |0\rangle \otimes |2\rangle \otimes |0\rangle \otimes \ldots, \\
||3\rangle\rangle &:= |0\rangle \otimes |0\rangle \otimes |3\rangle \otimes \ldots, \\
&\vdots
\end{aligned}
$$

So formally, can consider quantum walk on a line!

Set

$$
T_B = \sum_{k=m}^{\infty} ||k\rangle\rangle\langle\langle k||,
$$

a “hitting operator” in a time-of-arrival-type problem.

The model: map to random walk

Initial state and signal state:

$$\begin{aligned} |\Psi\rangle &= |0\rangle \otimes |0\rangle \otimes |0\rangle \otimes \ldots, \\ |\Psi_{\text{signal}}\rangle &= |1\rangle \otimes |0\rangle \otimes |0\rangle \otimes \ldots. \end{aligned}$$

Signal state couples only to

$$\begin{aligned} ||1\rangle\rangle &:= |1\rangle \otimes |0\rangle \otimes |0\rangle \otimes \ldots, \\ ||2\rangle\rangle &:= |0\rangle \otimes |2\rangle \otimes |0\rangle \otimes \ldots, \\ ||3\rangle\rangle &:= |0\rangle \otimes |0\rangle \otimes |3\rangle \otimes \ldots, \\ &\vdots \end{aligned}$$

So formally, can consider quantum walk on a line!

Set

$$T_B = \sum_{k=m}^{\infty} ||k\rangle\rangle\langle\langle k||,$$

a "hitting operator" in a time-of-arrival-type problem.

The model: map to random walk

Initial state and signal state:

$$|\Psi\rangle = |0\rangle \otimes |0\rangle \otimes |0\rangle \otimes \dots,$$
$$|\Psi_{\text{signal}}\rangle = |1\rangle \otimes |0\rangle \otimes |0\rangle \otimes \dots.$$

Signal state couples only to

$$||1\rangle\rangle := |1\rangle \otimes |0\rangle \otimes |0\rangle \otimes \dots,$$
$$||2\rangle\rangle := |0\rangle \otimes |2\rangle \otimes |0\rangle \otimes \dots,$$
$$||3\rangle\rangle := |0\rangle \otimes |0\rangle \otimes |3\rangle \otimes \dots,$$
$$\vdots$$

So formally, can consider quantum walk on a line!

Set

$$T_B = \sum_{k=m}^{\infty} ||k\rangle\rangle\langle\langle k||,$$

a "hitting operator" in a time-of-arrival-type problem.

The model: map to random walk

Initial state and signal state:

$$
\begin{aligned}
|\Psi\rangle &= |0\rangle \otimes |0\rangle \otimes |0\rangle \otimes \ldots, \\
|\Psi_{\text{signal}}\rangle &= |1\rangle \otimes |0\rangle \otimes |0\rangle \otimes \ldots.
\end{aligned}
$$

Signal state couples only to

$$
\begin{aligned}
||1\rangle\rangle &:= |1\rangle \otimes |0\rangle \otimes |0\rangle \otimes \ldots, \\
||2\rangle\rangle &:= |0\rangle \otimes |2\rangle \otimes |0\rangle \otimes \ldots, \\
||3\rangle\rangle &:= |0\rangle \otimes |0\rangle \otimes |3\rangle \otimes \ldots, \\
&\vdots
\end{aligned}
$$

So formally, can consider quantum walk on a line!

Set

$$
T_B = \sum_{k=m}^{\infty} ||k\rangle\rangle\langle\langle k||,
$$

a “hitting operator” in a time-of-arrival-type problem.

The model: effective Hamiltonian

In the $||k\rangle\rangle$-basis, Hamiltonian takes form

$$H = i \begin{pmatrix} 0 & -1 & & & & \\ 2 & 0 & -2 & & & \\ & 3 & 0 & -3 & & \\ & & 4 & 0 & -4 & \\ & & & \ddots & \ddots & \ddots \end{pmatrix}.$$

- Much simpler – but naive Schrödinger-picture solution still seems intractable.
- But: for suitable observables, Heisenberg-picture dynamics computable!

The model: effective Hamiltonian

In the $||k\rangle\rangle$-basis, Hamiltonian takes form

$$H = i \begin{pmatrix} 0 & -1 & & & & \\ 2 & 0 & -2 & & & \\ & 3 & 0 & -3 & & \\ & & 4 & 0 & -4 & \\ & & & \ddots & \ddots & \ddots \end{pmatrix}.$$

- Much simpler – but naive Schrödinger-picture solution still seems intractable.
- But: for suitable observables, Heisenberg-picture dynamics computable!

The model: effective Hamiltonian

In the $||k\rangle\rangle$-basis, Hamiltonian takes form

$$H = i \begin{pmatrix} 0 & -1 & & & & \\ 2 & 0 & -2 & & & \\ & 3 & 0 & -3 & & \\ & & 4 & 0 & -4 & \\ & & & \ddots & \ddots & \ddots \end{pmatrix}.$$

- Much simpler – but naive Schrödinger-picture solution still seems intractable.
- But: for suitable observables, Heisenberg-picture dynamics computable!

The model: Schrödinger picture time evolution

Introduce "position operator"

$$X = \sum_k k||k\rangle\rangle\langle\langle k||.$$

Key trick: algebra generated by commutators of X and H is small (in fact, isomorphic to $su(2)$).

Hence Heisenberg-dynamics $X(t)$ and

$$\langle\Psi_{\text{signal}}|X(t)|\Psi_{\text{signal}}\rangle = \frac{1}{2}(1+\cosh(2t)) \simeq \frac{1}{4}e^{2t}$$

explicitly computable.

The model: Schrödinger picture time evolution

Introduce "position operator"

$$X = \sum_k k||k\rangle\rangle\langle\langle k||.$$

Key trick: algebra generated by commutators of X and H is small (in fact, isomorphic to $su(2)$).

Hence Heisenberg-dynamics $X(t)$ and

$$\langle \Psi_{\text{signal}} | X(t) | \Psi_{\text{signal}} \rangle = \frac{1}{2}(1 + \cosh(2t)) \simeq \frac{1}{4} e^{2t}$$

explicitly computable.

The model: Schrödinger picture time evolution

Introduce "position operator"

$$X = \sum_k k||k\rangle\rangle\langle\langle k||.$$

Key trick: algebra generated by commutators of X and H is small (in fact, isomorphic to $su(2)$).

Hence Heisenberg-dynamics $X(t)$ and

$$\langle \Psi_{\text{signal}}|X(t)|\Psi_{\text{signal}}\rangle = \frac{1}{2}(1+\cosh(2t)) \simeq \frac{1}{4}e^{2t}$$

explicitly computable.

The model: bounding hitting time

Get lower bound on signal strengh:

Minimize

$$\delta = \mathrm{tr}[\rho(t)T]$$

Subject to

$$\begin{aligned} \langle X \rangle_{\rho(t)} &= \langle \Psi_{\mathrm{signal}} | X(t) | \Psi_{\mathrm{signal}} \rangle, \\ \langle X^2 \rangle_{\rho(t)} &= \langle \Psi_{\mathrm{signal}} | X^2(t) | \Psi_{\mathrm{signal}} \rangle. \end{aligned}$$

- That's an infinite-dim. convex optimization problem (obvious).
- Solve by passing to Lagrange dual, guess solution for dual, ...(not so obvious).

The model: bounding hitting time

Get lower bound on signal strengh:

Minimize

$$\delta = \mathrm{tr}[\rho(t)\, T]$$

Subject to

$$\begin{aligned} \langle X \rangle_{\rho(t)} &= \langle \Psi_{\mathrm{signal}} | X(t) | \Psi_{\mathrm{signal}} \rangle, \\ \langle X^2 \rangle_{\rho(t)} &= \langle \Psi_{\mathrm{signal}} | X^2(t) | \Psi_{\mathrm{signal}} \rangle. \end{aligned}$$

- That's an infinite-dim. convex optimization problem (obvious).
- Solve by passing to Lagrange dual, guess solution for dual, ...(not so obvious).

The model: bounding hitting time

Get lower bound on signal strengh:

Minimize

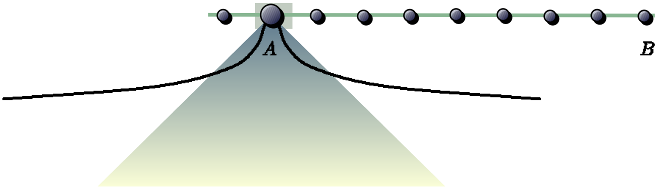
$$\delta = \mathrm{tr}[\rho(t)\,T]$$

Subject to

$$\begin{aligned} \langle X \rangle_{\rho(t)} &= \langle \Psi_{\mathrm{signal}} | X(t) | \Psi_{\mathrm{signal}} \rangle, \\ \langle X^2 \rangle_{\rho(t)} &= \langle \Psi_{\mathrm{signal}} | X^2(t) | \Psi_{\mathrm{signal}} \rangle. \end{aligned}$$

- That's an infinite-dim. convex optimization problem (obvious).
- Solve by passing to Lagrange dual, guess solution for dual, …(not so obvious).

Results



- Signal strength:
$$\delta \geq 1/5,$$
independent of distance m;
- At time
$$t = \log m,$$
logarithmic in distance.

Summary

We have...

- ...exhibited models which allow for exponentially accelerating excitations,
- ...highlighted the non-triviality of Lieb-Robinson bounds,
- ...shown that any algorithm for the simulation of such models must deal with far away regions exchanging information at short time scales.

Thank you for your attention.

J. Eisert and D. Gross, arXiv:0808.3581.

Universität Potsdam

CAROLO-WILHELMINA BRAUNSCHWEIG

Imperial College London