Most quantum states are useless for measurement-based quantum computation

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Measurement-based QC

Raussendorf & Briegel PRL 2001
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$|+\rangle$  $|+\rangle$
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Without initial entanglement, it’s clear you can’t do better than BPP.
Universality and entanglement

Question:
What are the necessary and sufficient conditions for a family of $n$ qubit quantum states to be universal for MBQC?
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Necessary conditions:
van den Nest, Miyake, Dür, Briegel 2006
find entanglement measures that must grow “quickly” with $n$. 
Universality and entanglement

Question:
What are the necessary and sufficient conditions for a family of n qubit quantum states to be universal for MBQC?

Sufficient conditions:
Gross, Eisert, Schuch, Pérez-García 2007
find states with special structure in the many-body correlations.
find ground states with special structure.
Bridging the divide

Quantum world

Classical world
Bridging the divide

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Entanglement and correlations

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Local bases, Limited processing power.
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MBQC
Local bases, geometric measure

\[ E_g(\Psi) = -\log_2 \sup_{\alpha \in \mathcal{P}} |\langle \alpha | \Psi \rangle|^2 \]

the set of product states

Answers the question: How far is the nearest collection of local bases \( \alpha_1, \alpha_2, \ldots, \alpha_n \)?

Large geometric measure

Far from all product states
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Theorem 1 (GFE): \( n \) qubit states with \( E_g > n - O(\log n) \) are useless for MBQC.
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For concreteness, a state is useless if it fails to provide a polynomial-time MBQC algorithm for Factoring.

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For concreteness, a state is **useless** if it fails to provide a polynomial-time MBQC algorithm for Factoring.

Proof strategy: replace $\psi$ with a classical coin and show there exists a classical algorithm that factors just as well (within poly factors).

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\[ |\Psi\rangle \]
bases:
α₁

| Ψ ⟩
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$$\alpha_1$$
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$0110$
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$$|\langle \alpha | \Psi \rangle|^2 \leq 2^{-E_g} \leq 2^{-n + \delta}$$

$$\Rightarrow \frac{|G|}{2^n} > 2^{-\delta - 1} = \text{poly}(1/n).$$
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To simulate classically, just ignore the measurement results and use a classical coin!
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Useless for MBQC
Large geometric measure ➔ Useless for MBQC

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\[ E_g < n - O(\log n) \]
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Theorem 2 (GFE): In fact, they are abundant. This is vacuous unless such states exist.

The proof involves standard measure concentration arguments (via $\epsilon$-nets) and known results about random states.
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d-level systems

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$$V : \mathbb{C}^d \rightarrow \mathbb{C}^d \otimes \mathbb{C}^d$$

$$V |\beta\rangle = U |0\rangle \otimes |\beta\rangle$$
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Concatenate to get the state of $2^k$ qudits at level $k$.

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Decision problems

I have a generic $\Psi$. Can I compute anything with it?

For almost every state $\Psi$, there is no poly-bounded classical control circuit which allows a significant advantage over classical randomness. Only problems in BPP can be solved. (BMW ‘08)

$$\Pr_{\Psi} \{ \exists C \ | C(\Psi) - C(2^{-n}1) | > \epsilon \} \leq (8^8 w)^{3v} e^{-c\epsilon^2 2^n}$$
Randomness vs entanglement?

Random states such that $E_g \leq \log K + O(1)$ also offer no advantage!

- Choose $nK$ states at random from $C^2$ to construct the following (where $K$ is superpolynomial in $n$):

$$R := \sum_{j=1}^{K} |\psi_j^{(1)}\rangle\langle\psi_j^{(1)}| \otimes \cdots \otimes |\psi_j^{(n)}\rangle\langle\psi_j^{(n)}|$$

- Randomly pick a state from the support of $R$ then:

$$|\Psi\rangle = \frac{1}{\sqrt{\langle \Psi_0 | R | \Psi_0 \rangle}} \sqrt{R} |\Psi_0\rangle$$

$$\Pr_{\Psi} \{ \exists C \mid |C(\Psi) - C(2^{-n}1)| > \epsilon \} \leq \left( 2^n + (8^8w)^{3v} \right) e^{-c'\epsilon^2K^{1/3}}$$
Questions

- Can we derandomize these constructions?
- Can Hastings’ techniques give improved bounds?
- Are efficiently created states subject to this effect?
- What happens with a polynomial number of copies?
- What implications does this have for the circuit model?