

# Ground State Entanglement in 1-Dimensional Translationally-Invariant Quantum Systems

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[arXiv:0901.1107](https://arxiv.org/abs/0901.1107)

# Introduction

- What are the minimal set of properties a Hamiltonian must have in order for its ground state to have a high degree of entanglement?
- Do symmetries such as translational-invariance limit entanglement?

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- What are the minimal set of properties a Hamiltonian must have in order for its ground state to have a high degree of entanglement?
- Do symmetries such as translational-invariance limit entanglement?
  - One-dimensional systems.
  - Entropy of entanglement when traced down to a contiguous region in the 1D chain.

# Previous Work

- Local Hamiltonian is QMA-complete, even for 1D systems.
- Adiabatic computation on 1D systems can perform universal quantum computation [AIGK07]
  - Can these constructions be made translationally-invariant?
    - Translationally invariant modification that can be used for 1D universal adiabatic computation.  
[Nagaj-Wocjan, Janzin-Wocjan-Zhang]
      - » Degenerate
    - 1D Local Hamiltonian is QMA-complete, even what all two-particle terms are the same. [Kay]
      - » Requires position-dependent 1-particle terms.

# Translation-Invariance

- Can we make the 1D construction translationally-invariant with a Hamiltonian which has a non-degenerate ground state?
- If the system is described by a single Hamiltonian term applied to all pairs of particles (with bounded precision), how do we encode a circuit?
  - Show high entanglement in the ground state.
  - For 1D systems, previously known bounds, ground state entanglement entropy scales logarithmically with the region size.

# Results

- Finite 1D chain:
  - Single Hamiltonian term  $H$  that operates on two particles of dimension  $2^1$ , when applied to every neighboring pair in a finite chain of  $n$  particles:
    - Unique ground state
    - Spectral gap  $1/\text{poly}(n)$
    - The entropy of entanglement of a region of size  $m$  on either end of the chain is  $\mathcal{W}(\min\{m, n-m\})$ .

# Results

- Cycles and the Infinite Chain:
  - Family of translationally-invariant Hamiltonians  $\{H_n\}$  for a cycle of  $nt$  21-dimensional particles
    - Spectral gap is  $1/\text{poly}(n)$
    - For any state in the ground space of  $H_n$ , and any  $m$ , there exist regions of size of  $m$  whose entanglement entropy is  $\mathcal{W}(\min\{m,n\})$ .
      - Entanglement bounds for a constant fraction of regions of size  $m$ .
  - Bounds hold in the limit as  $t$  tends towards infinity.

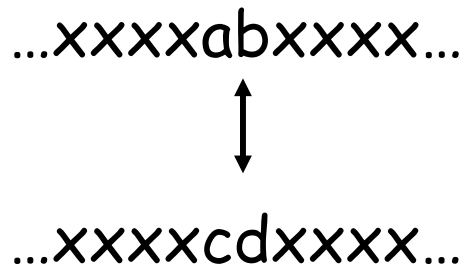
# 1D Area Law

- Upper bound on the entanglement entropy for any ground state of a 1D Hamiltonian  $H$ , independent of region size but exponentially dependent on  $1/D$ , where  $D$  is the spectral gap of  $H$ . [Hastings 07]
- Gottesman and Hastings: is the dependence on  $1/D$  tight?
  - Family of 1D Hamiltonians with unique ground state with regions whose entanglement entropy is  $\mathcal{W}(\text{poly}(1/D))$ .
  - Previously, best known such lower bound was  $\mathcal{W}(\log(1/D))$ 
    - [arXiv:0901.1108](#)
- Construction present here gives a similar result.



# Hamiltonian Construction Basics

- Type I terms (illegal pairs)
  - $|ab\rangle\langle ab|$
  - Energy penalty for:  $\dots xxxabxxxx\dots$
- Type II terms (transition rules)
  - $\frac{1}{2}(|ab\rangle\langle ab| + |cd\rangle\langle cd| - |ab\rangle\langle cd| - |cd\rangle\langle ab|)$



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$\dots xxxabxxxx\dots$



$\dots xxxcdxxxx\dots$

$ab \rightarrow cd$

# Hamiltonian Construction Basics

$|\psi_1\rangle$

↓

$|\psi_2\rangle$

↓

$|\psi_3\rangle$

⋮

↓

$|\psi_T\rangle$

Each  $|\psi_i\rangle$  contains no illegal pairs.

$$\text{Ground State} = \frac{1}{\sqrt{T}} \sum_{i=1}^T |\psi_i\rangle$$

[Kitaev02]

# One Dimensional Hamiltonians

- Two types of states:
  - Control states:  $\uparrow \rightarrow \leftarrow \downarrow \Rightarrow \Leftarrow$
  - Passive States:  $W w E e U u < >$
- Transition rules apply to control state and a state to the right or left.
  - May move control state to the left or right.
  - For example:  $(w) (\leftarrow) \dashrightarrow (\leftarrow) (W)$

[AGIK07]

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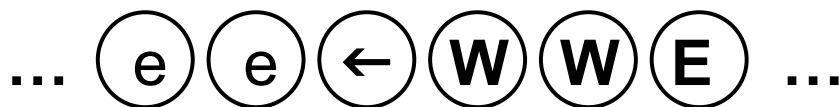
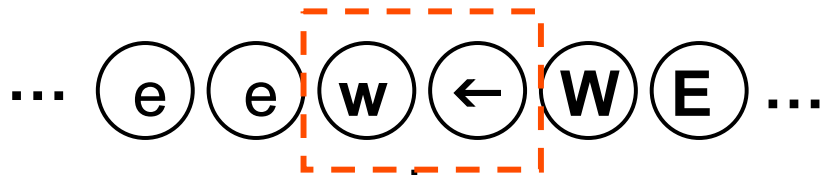
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[AGIK07]

# Construction Overview

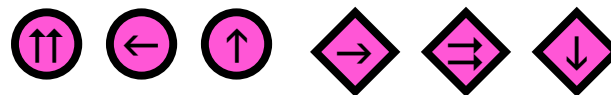
Circles: Single states



Diamonds: Two-dimensional subsystems



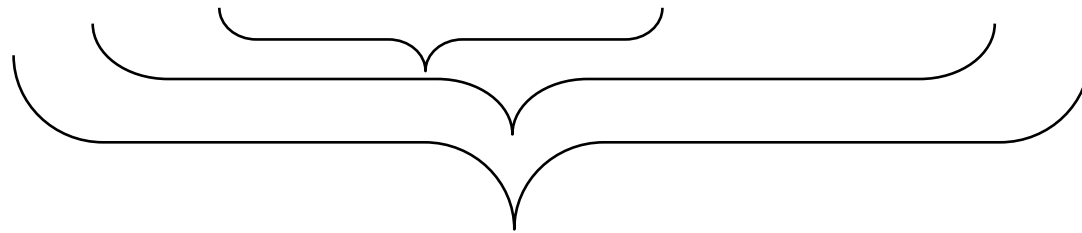
Control states:



Standard basis: specify state type for each site  
And then 0 or 1 for each 2D subsystem



# Construction Overview



EPR Pairs  $\frac{1}{\sqrt{2}}|00\rangle + \frac{1}{\sqrt{2}}|11\rangle$

$e_+$   $E_1$  Entangled states

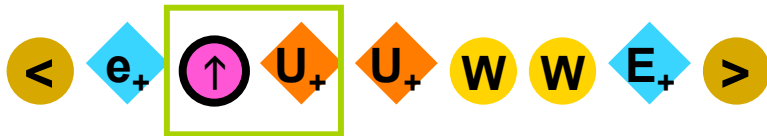
$u_0$   $u$  Unentangled states

$w$   $W$  Waiting states

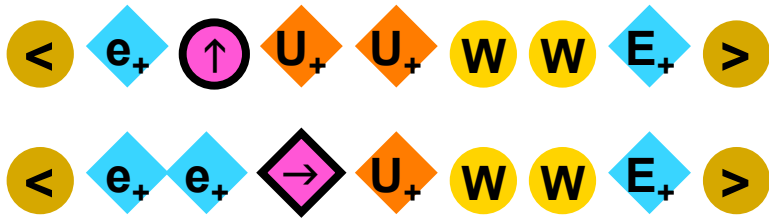
# Target Ground State



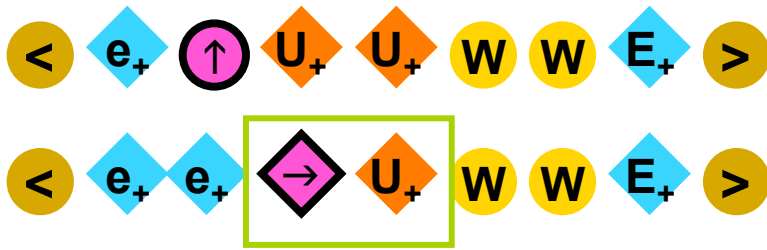
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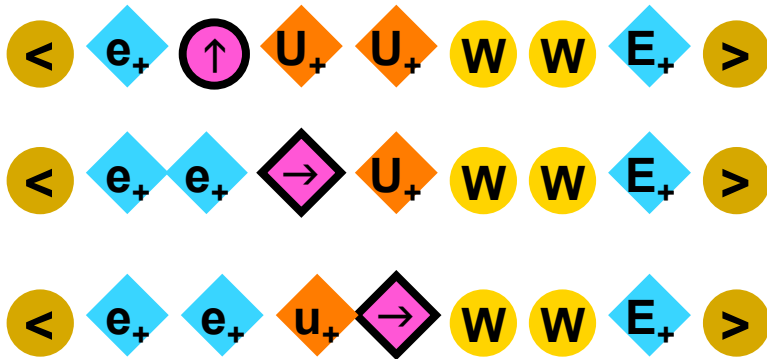
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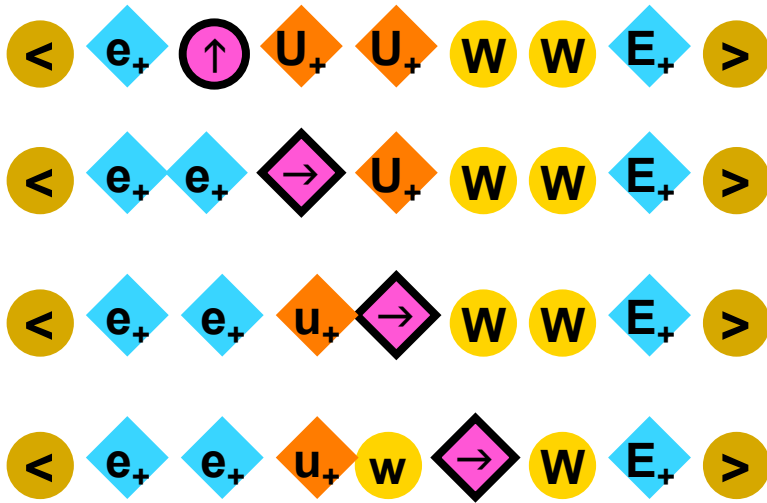
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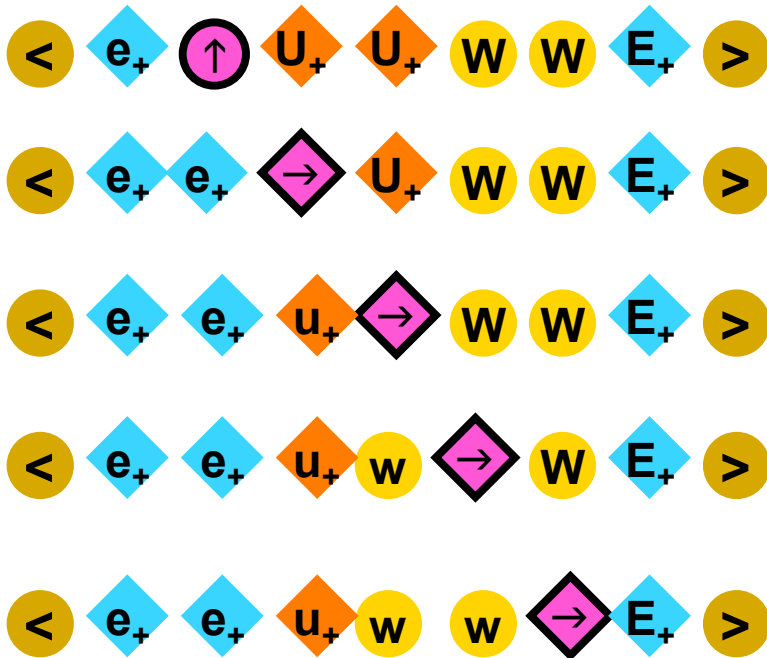
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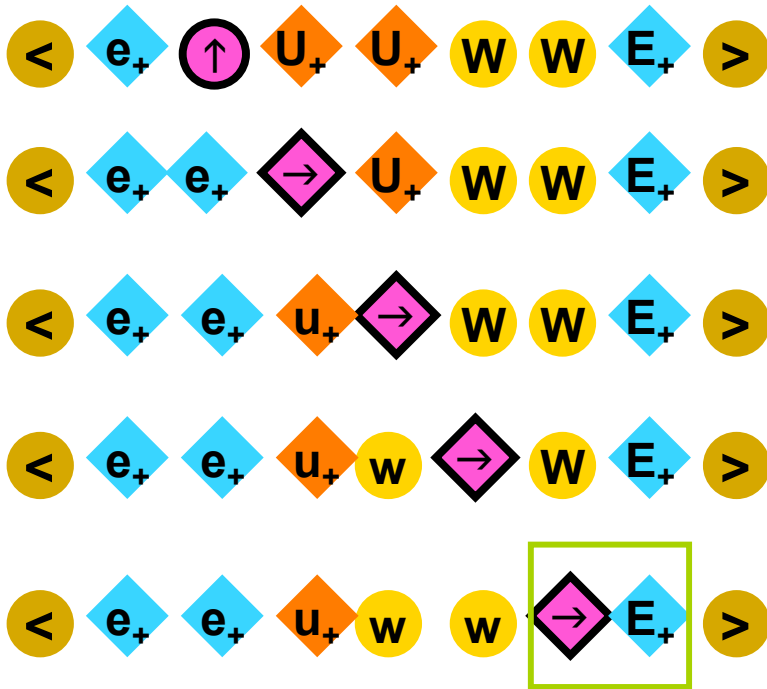


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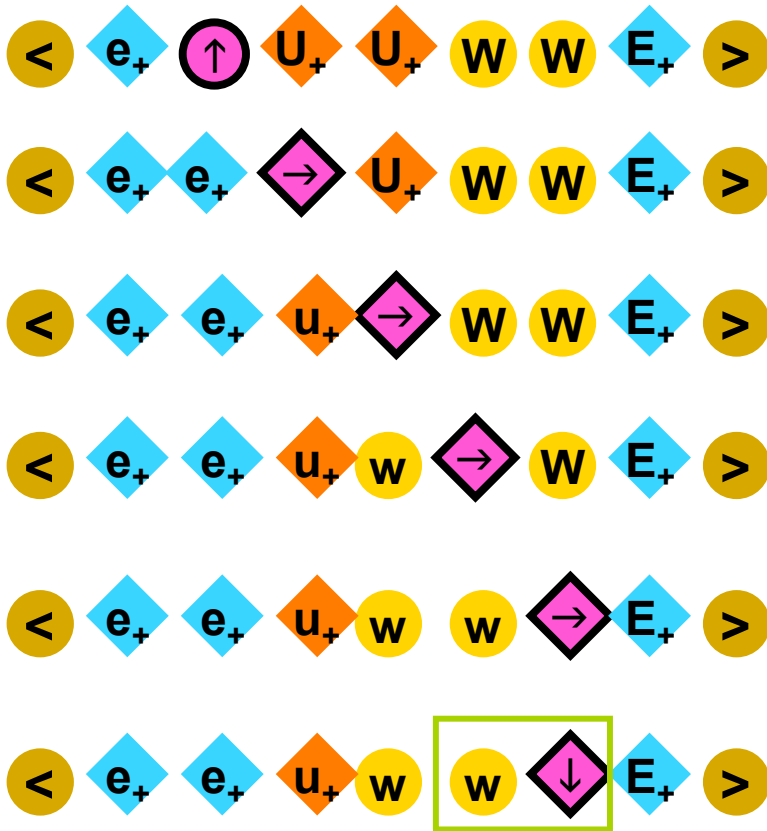




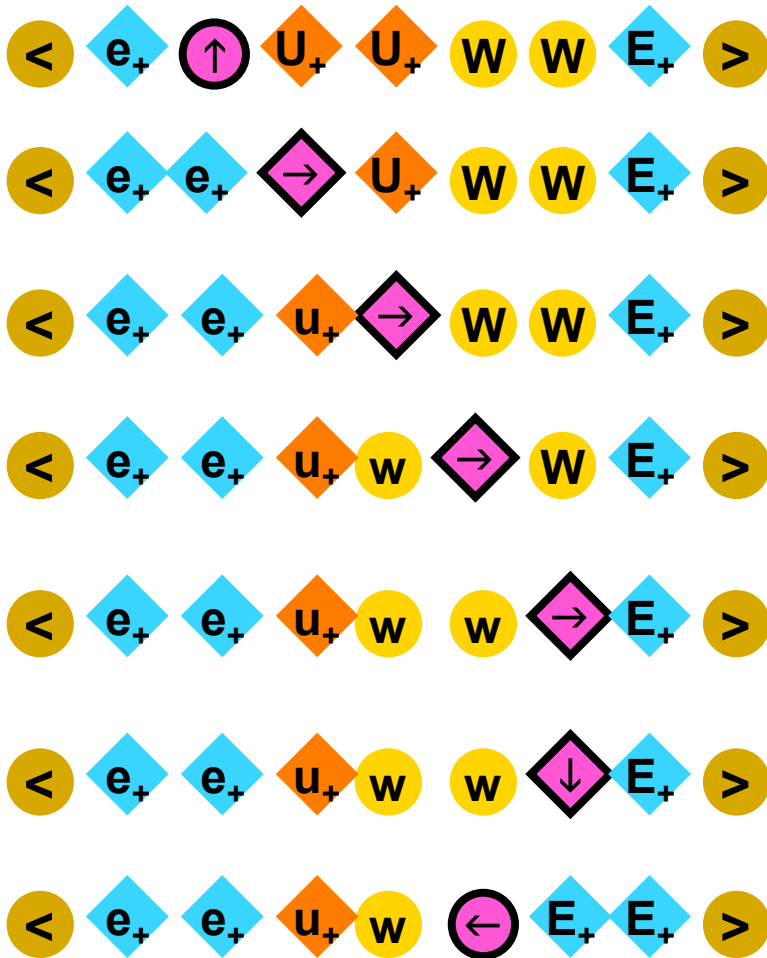
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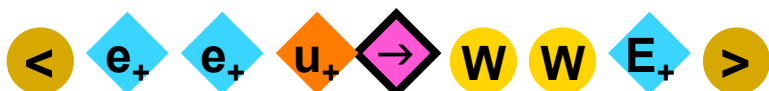
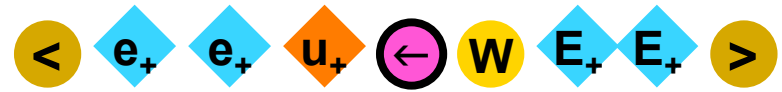
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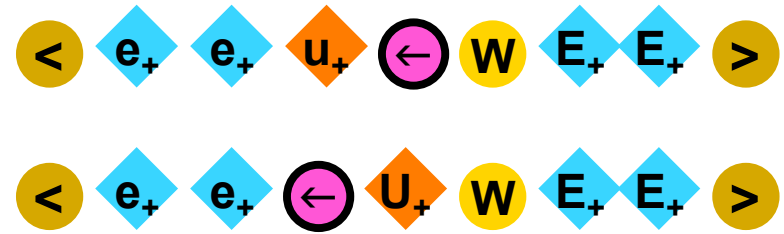
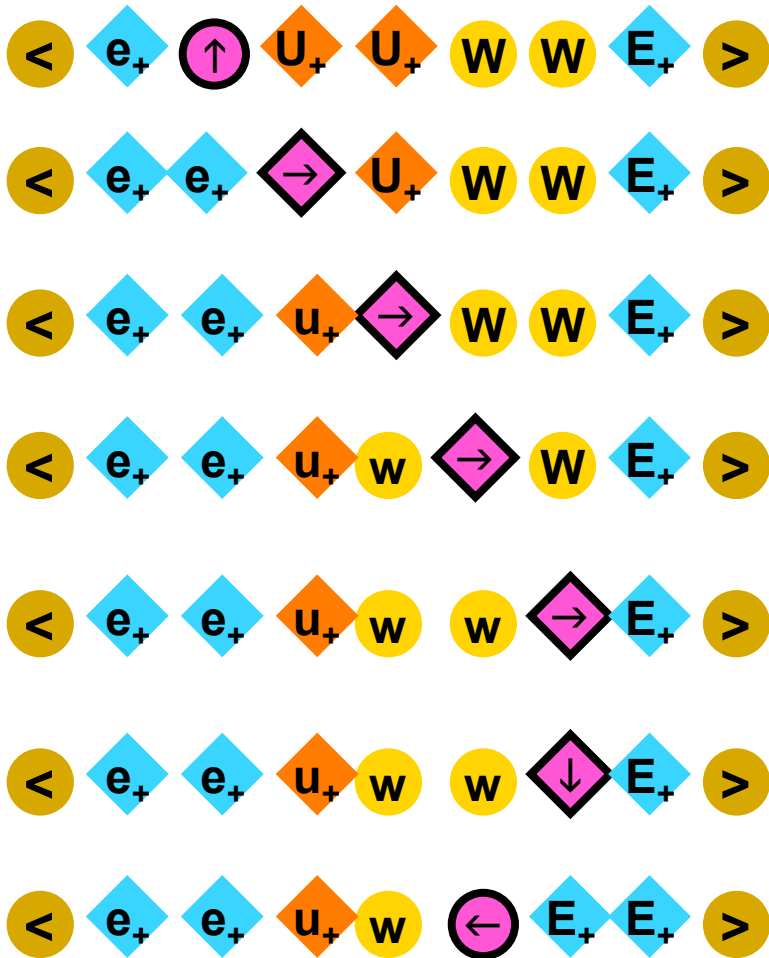
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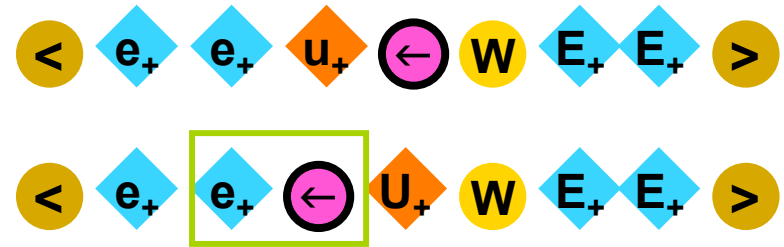
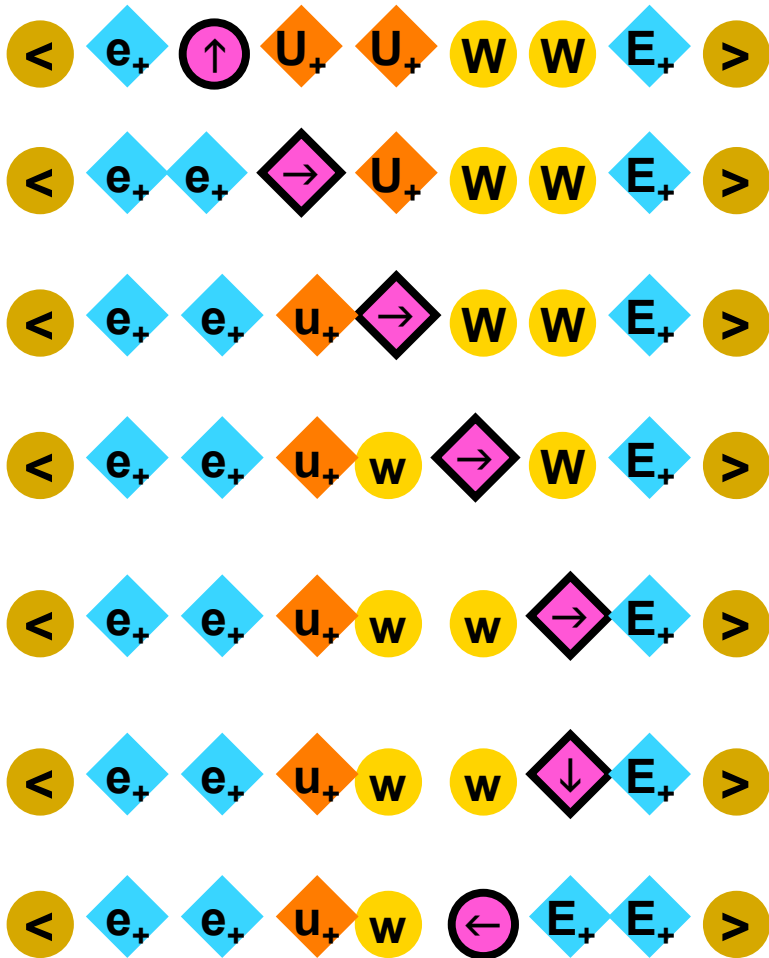
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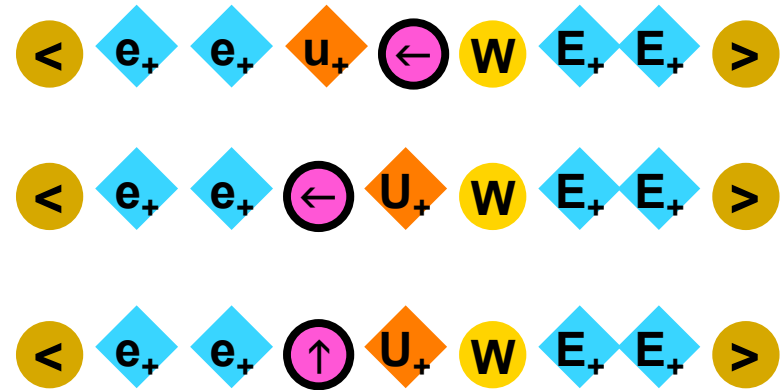
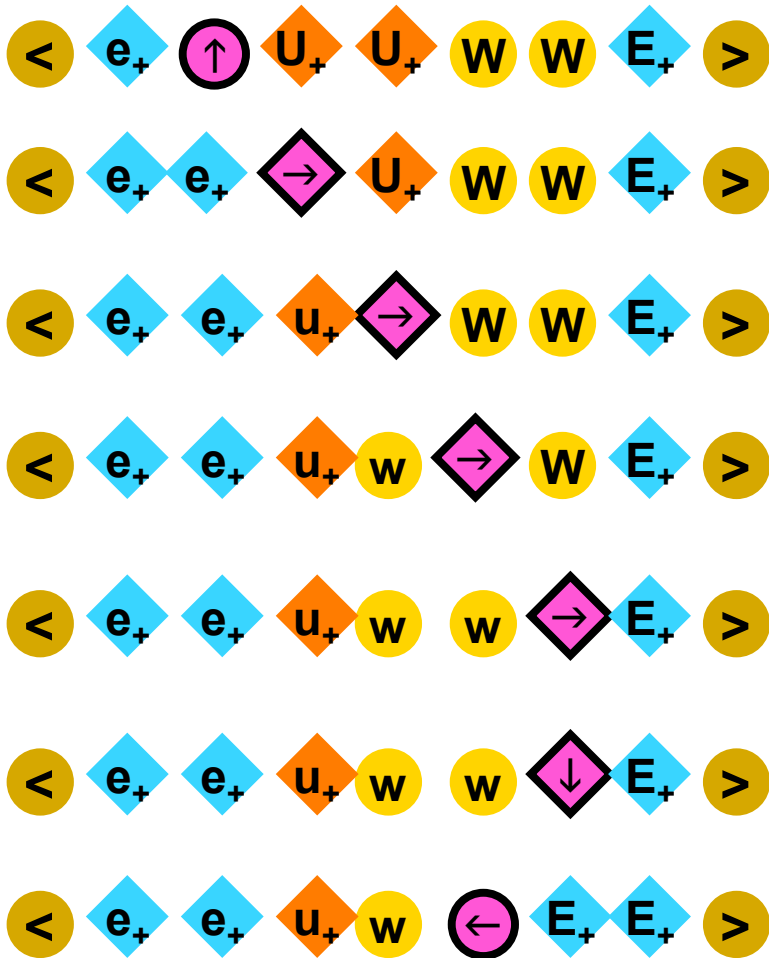
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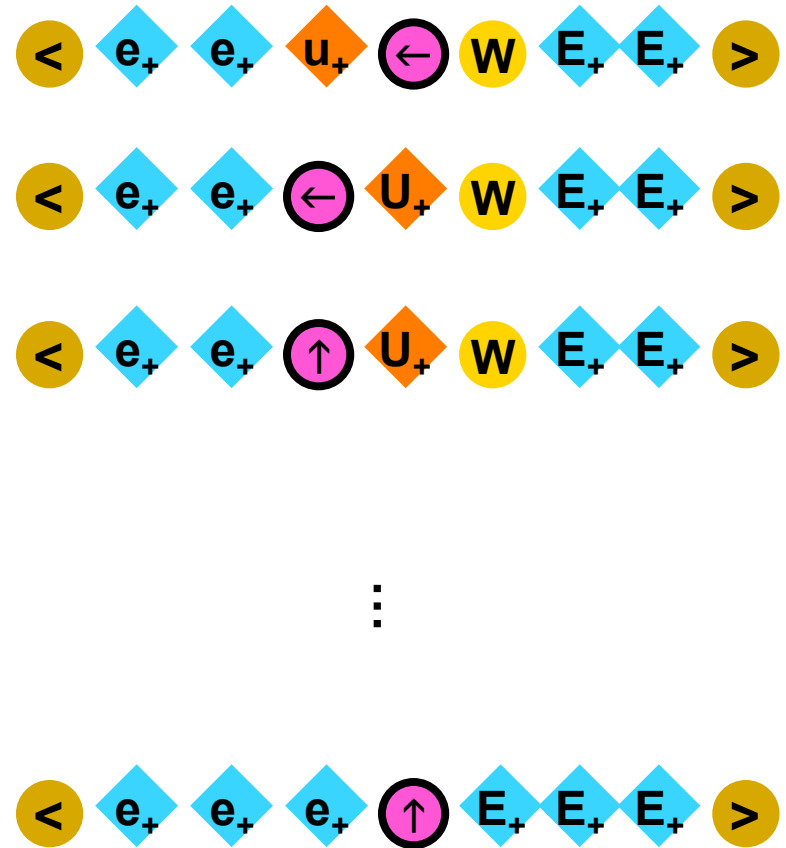
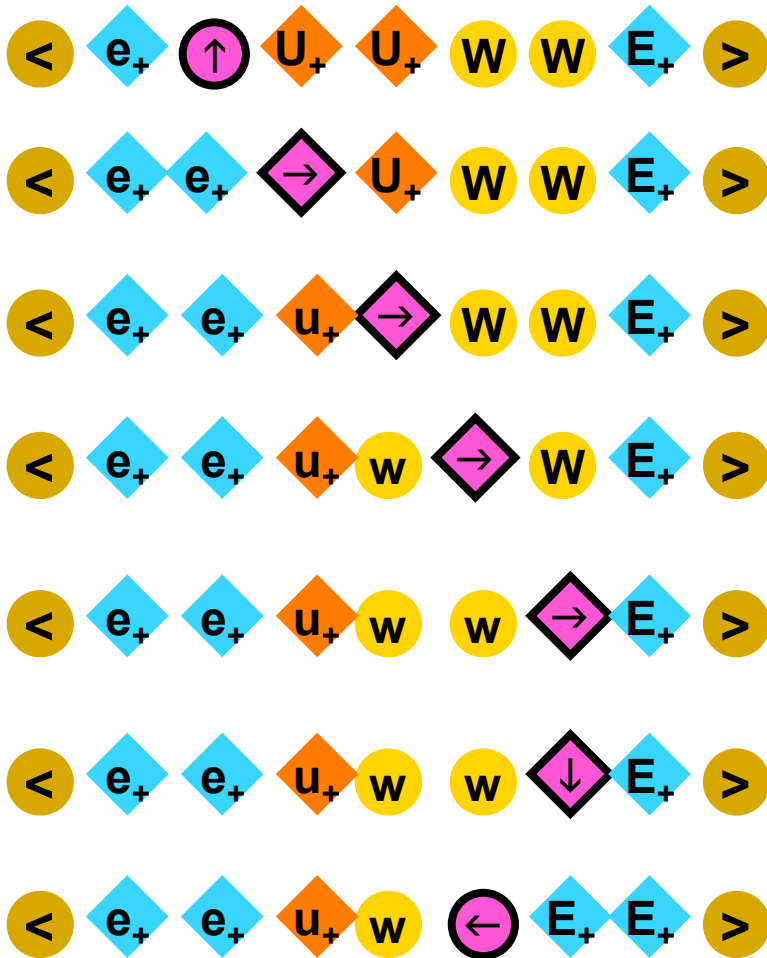
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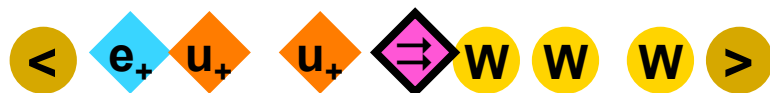


# Target Ground State





# First Round



Special control states for the first round force the left end and the right end to agree that it is the first round.

Will be used to check that state is symmetric about center.

# The Hamiltonian...so far

- $H_{\text{trans}}$  = sum of terms from transition rules as applied to all neighboring pairs of particles.
- $H_{\text{legal}}$  = sum of terms from illegal pairs
- $H = H_{\text{trans}} + H_{\text{legal}}$

# Well-Formed States

- A state in the standard basis is **well-formed** if it is of the form:
  - $(\langle +e)(\mathbf{e})^*(\mathbf{w+u})^*(\rightarrow+\Rightarrow+\leftarrow)(\mathbf{W+U})^*(\mathbf{E})^*(\rangle +e)$
  - $(\langle +e)(\mathbf{e})^*(\uparrow+\Uparrow)(\mathbf{W+U})^*(\mathbf{E})^*(\rangle +e)$
  - $(\langle +e)(\mathbf{e})^*(\mathbf{w+u})^*(\downarrow)(\mathbf{E})^*(\rangle +e)$
- ...or any substring of these forms
- Can be checked by local checks

# Well-Formed States

- Can not enforce by local checks that a legal state has a control state.
  - E.g.,  $eeee\dots eee$  must be a legal state.
- If the state is *bracketed* ( a  $\langle$  on left end and a  $\rangle$  on the right end), then a legal state must be exactly on of the following three forms:
  - $(\langle +e)(e)^*(w+u)^*(\rightarrow + \Rightarrow + \leftarrow)(W+U)^*(E)^*(\rangle +e)$
  - $(\langle +e)(e)^*(\uparrow + \Uparrow)(W+U)^*(E)^*(\rangle +e)$
  - $(\langle +e)(e)^*(w+u)^*(\downarrow)(E)^*(\rangle +e)$

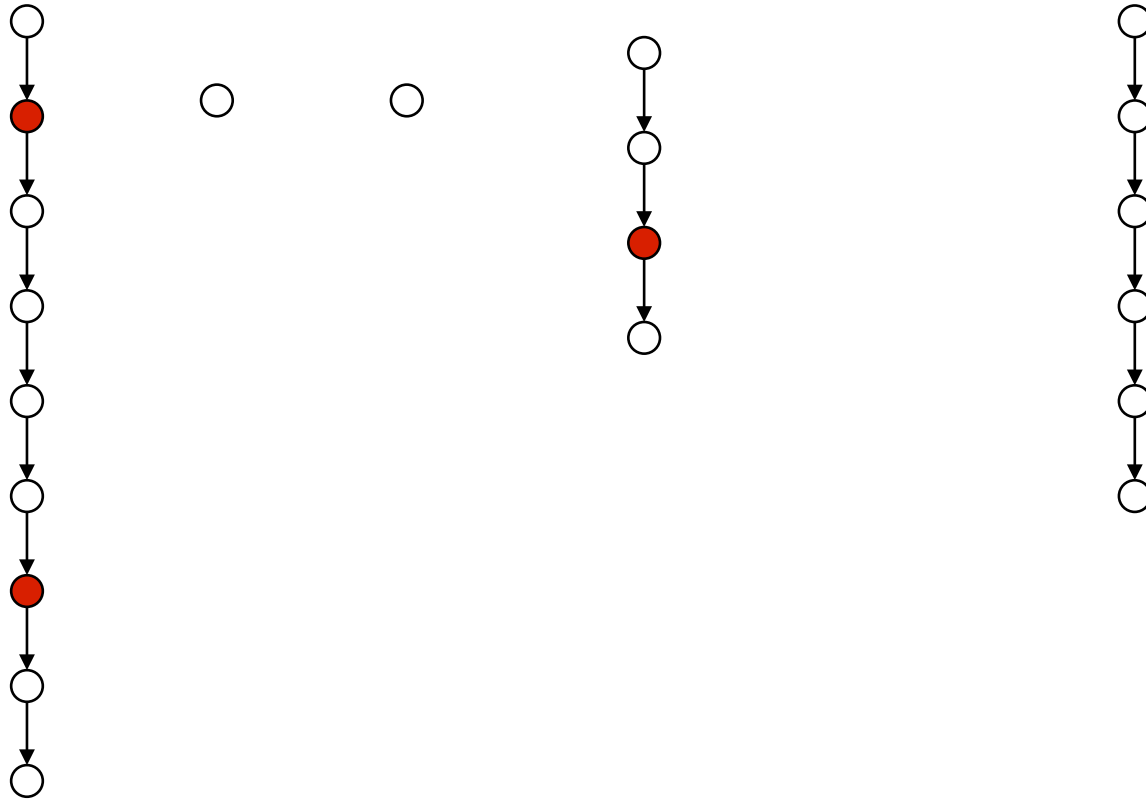
# Properties of Well-Formed States

- $H_{\text{trans}} + H_{\text{legal}}$  is closed over the subspace spanned by well-formed states.
  - (All additional terms will be diagonal in standard basis).
- For each well-formed state, at most one transition rule applies in the forward direction and at most one transition rule applies in the reverse direction.

# State Graph

- Nodes - set of all state in standard basis.
- Edges - directed edge from state  $A$  to state  $B$  if  $B$  can be obtained by applying one transition rule to  $A$ .
  - Well-formed states are disconnected from the rest of the graph.
  - State graph restricted to well-formed states form disjoint paths.

# Hamiltonians Restricted to Paths



# The Hamiltonian restricted to a single path

$$H_{legal} = \begin{bmatrix} x_1 & 0 & \cdots & 0 \\ 0 & x_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & x_l \end{bmatrix} \quad H_{trans} = \begin{bmatrix} \frac{1}{2} & -\frac{1}{2} & 0 & \cdots & 0 \\ -\frac{1}{2} & 1 & -\frac{1}{2} & 0 & \ddots & \vdots \\ 0 & -\frac{1}{2} & 1 & -\frac{1}{2} & 0 & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\ \vdots & 0 & -\frac{1}{2} & 1 & -\frac{1}{2} & 0 \\ & & 0 & -\frac{1}{2} & 1 & -\frac{1}{2} \\ 0 & \cdots & 0 & -\frac{1}{2} & \frac{1}{2} \end{bmatrix}$$

- If  $H_{legal}$  has a non-zero entry, the minimum eigenvalue of  $H$  restricted to the subspace spanned by states in the path is  $\Omega(1/l^3)$  which is  $\Omega(1/n^6)$ . [Kitaev 02]
- If  $H_{legal}$  is all zero, the state which is the uniform superposition of states in the path has zero energy.



# Bad States

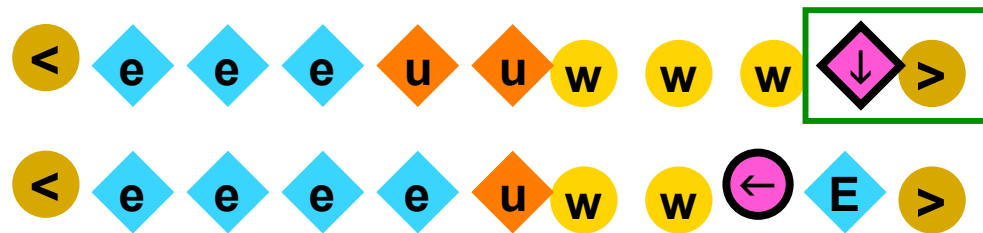


- Will check for these states by showing that they evolve (via forward or backward application of the transition rules) to illegal states.

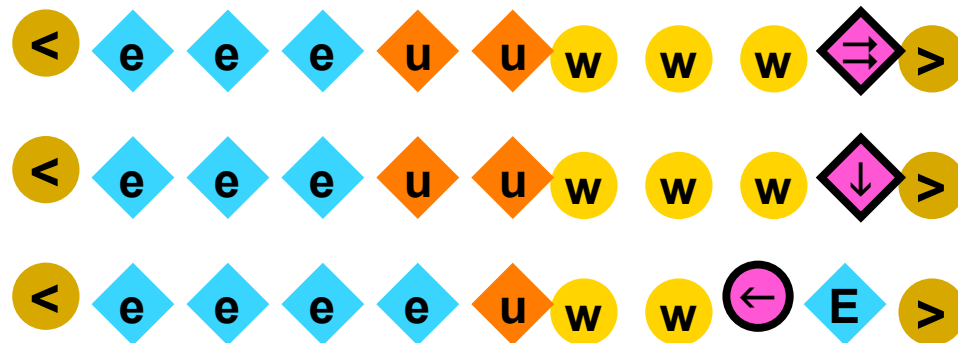
# How To Check for Bad States: An Example



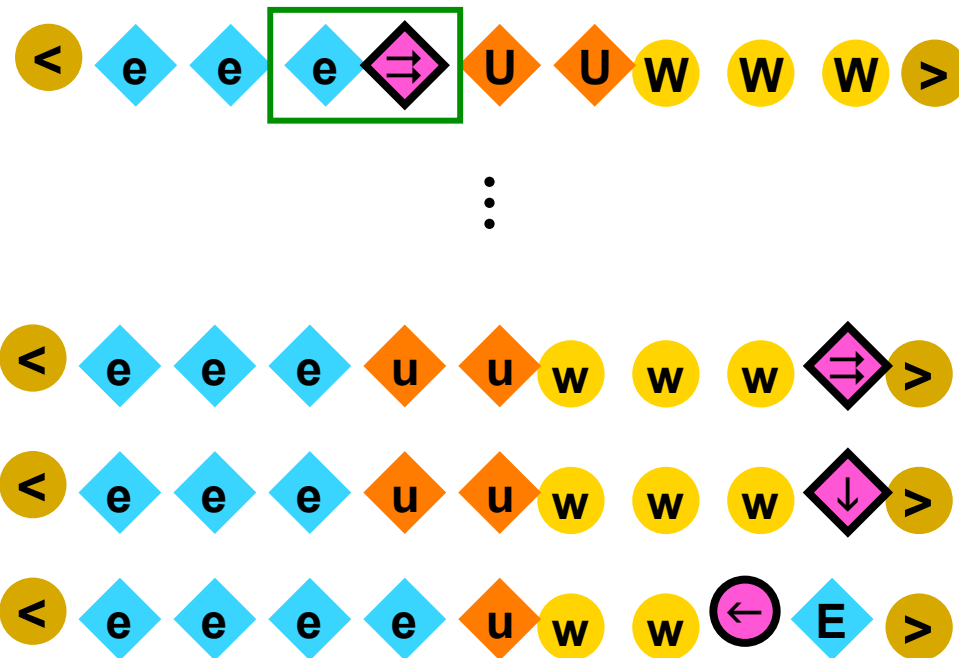
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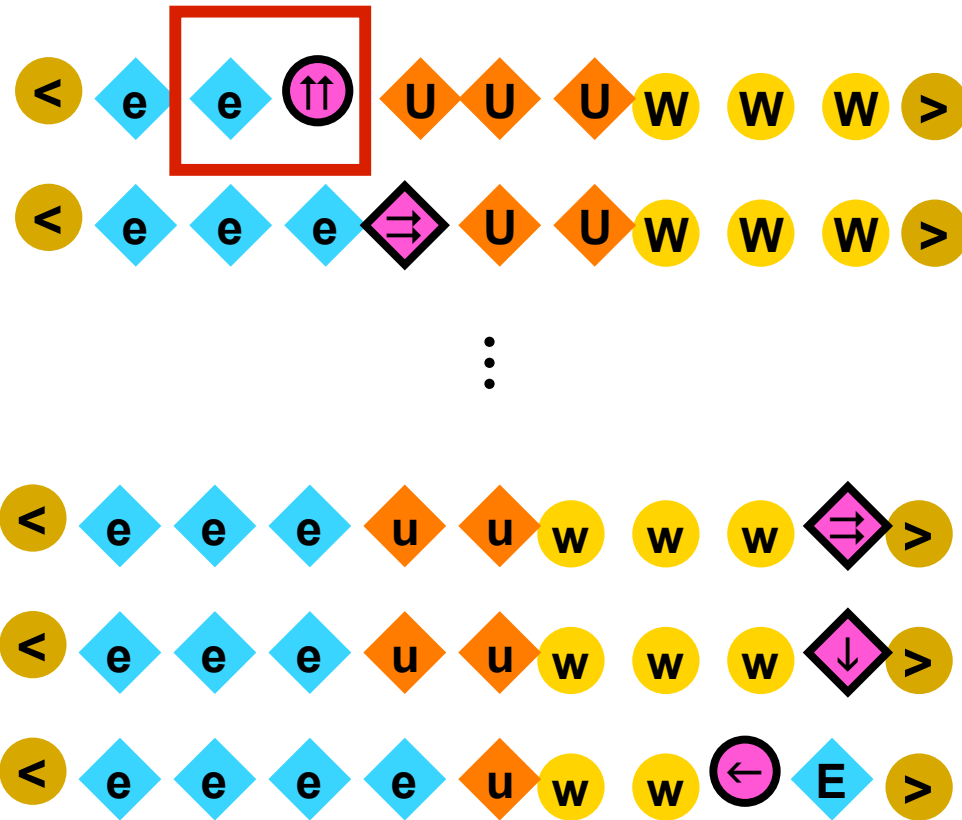
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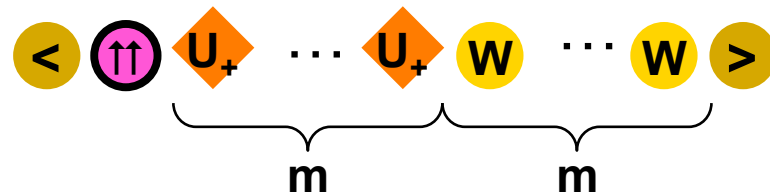


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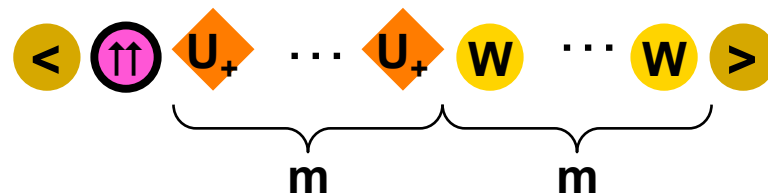
# Need to Show:

- A path that starts with



corresponds to a zero energy state for H.

- A path that does not start with

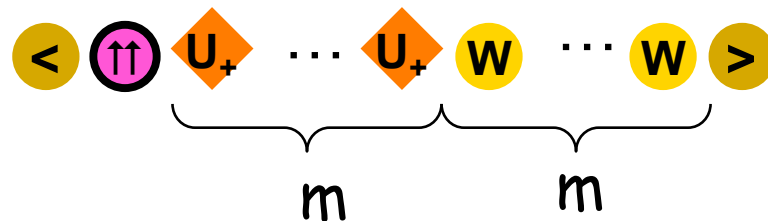


contains a state that has an illegal pair

OR does not have states that are bracketed. (Start with < and end with > ).

# Initializing Qubits

- $H_{\text{init}} = |u_- \rangle \langle u_-|$  Penalty for state  $U_-$
- Ensures that ground state corresponds to a path whose initial state has qubits set to  $|+\rangle$





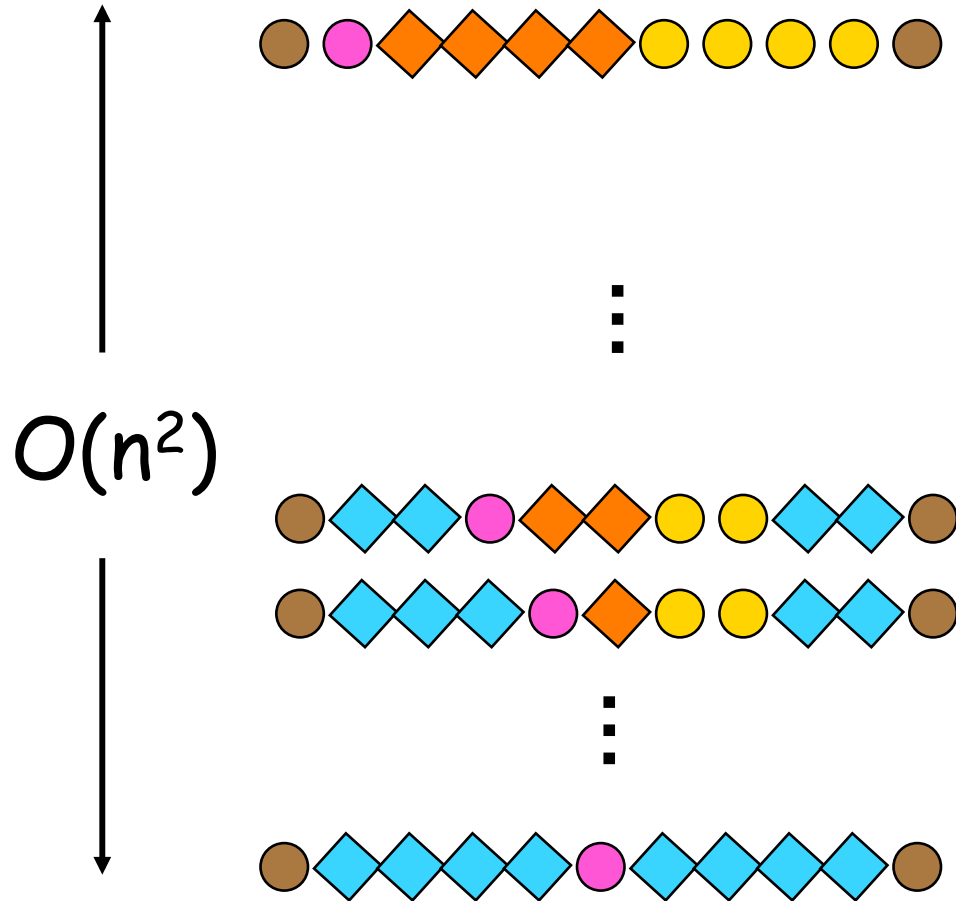
# Enforcing Bracketed States

$$H_{\text{bracket}} = I - | \langle \rangle \langle \rangle \langle \rangle | - | \rangle \rangle \rangle |$$

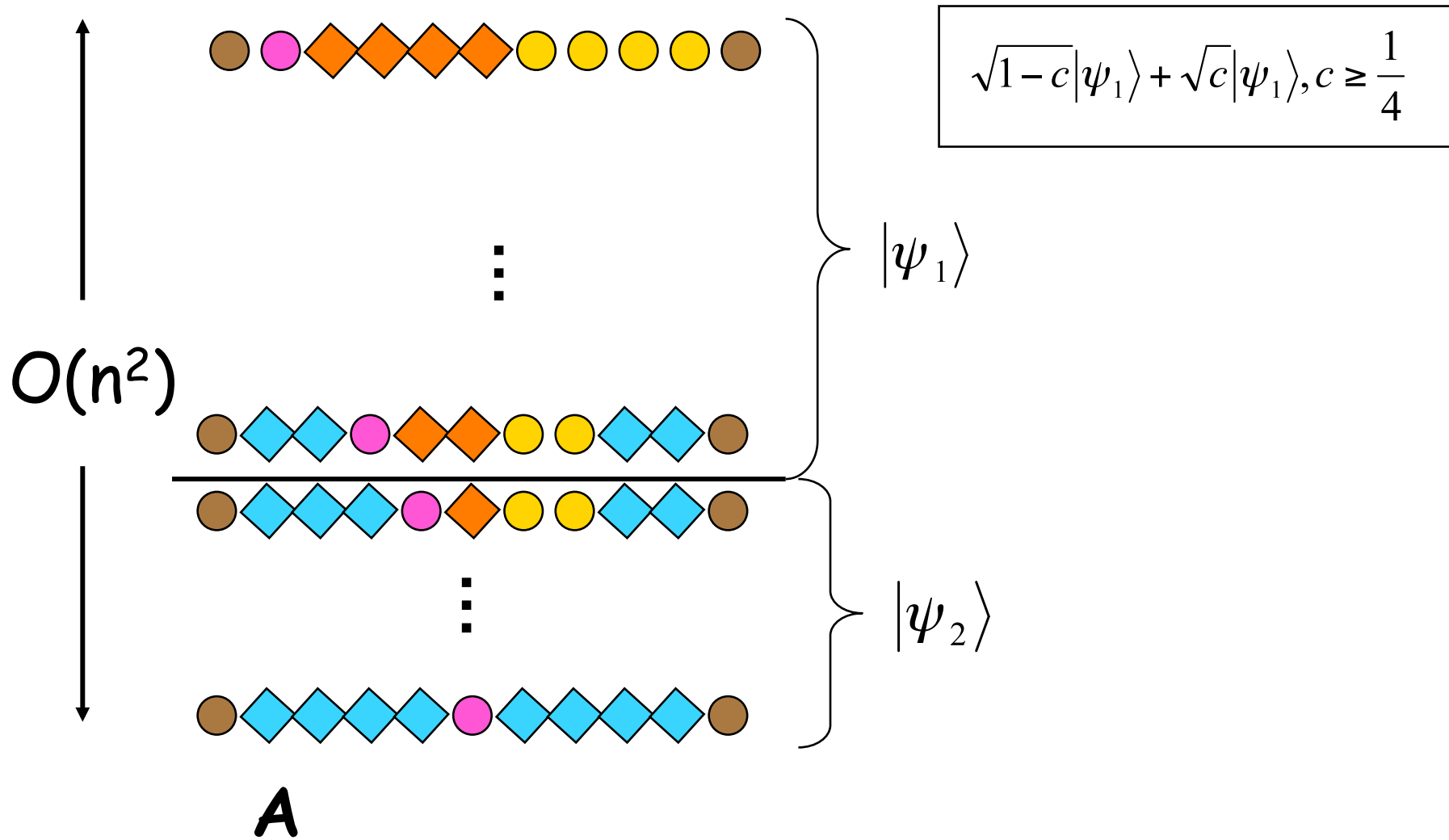
$$H = 3( H_{\text{trans}} + H_{\text{legal}} ) + H_{\text{init}} + H_{\text{bracket}}$$

- $3( H_{\text{trans}} + H_{\text{legal}} )$  term ensures there are no bracket terms in the middle.
- $H_{\text{bracket}}$  term gives an energy benefit for having brackets at the end

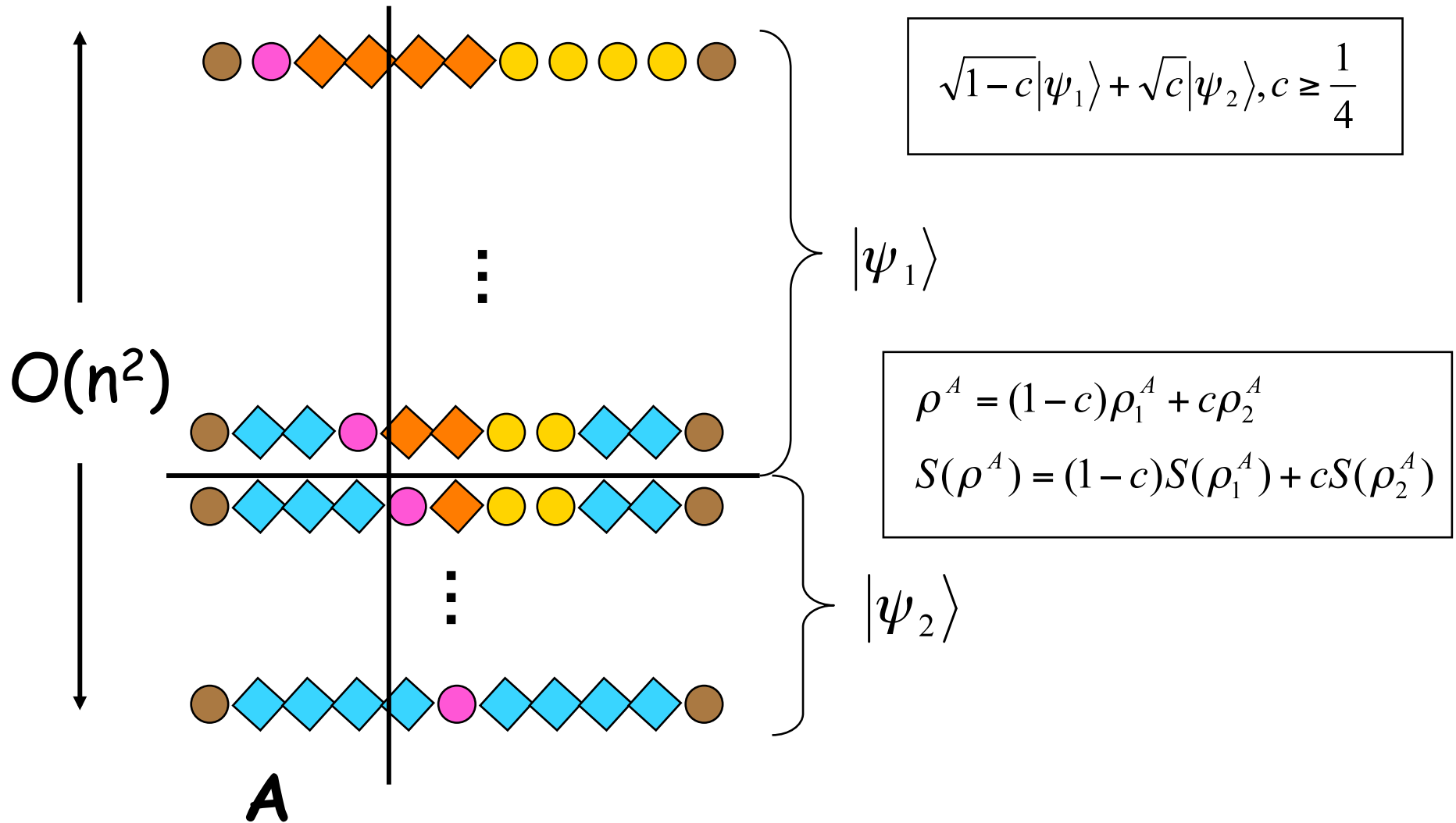
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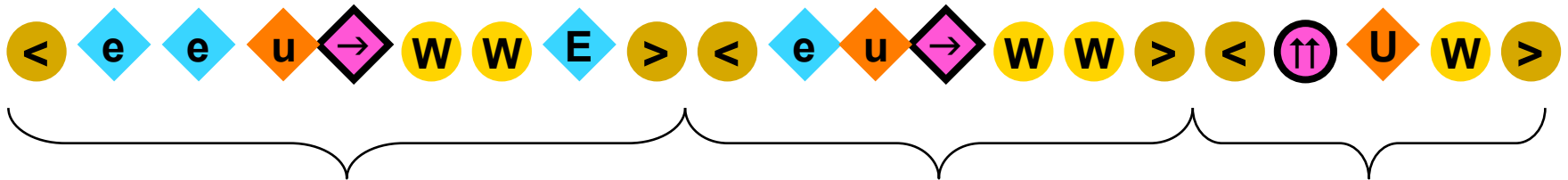


# Entropy of Entanglement



# Finite Cycle of size $tn$

- Change  $H_{\text{legal}}$  so that the pair  $> <$  is allowed.
- Well-formed states look like:



- A sequence from a  $<$  to a  $>$  is a *segment*.
- $H$  is closed on the set of well-formed states for a fixed set of segments.

## Finite Cycle Cont.

- $H = p(n)(H_{\text{legal}} + H_{\text{trans}} + H_{\text{init}}) + H_{\text{size}}$
- For  $p(n)$  large enough, using the Projection Lemma of Kempe-Kitaev-Regev, we can assume that the ground state of  $H$  is composed of tensor products of ground states for finite chains.
- Ground state for finite chain of length  $l$  is  $|\phi_l\rangle$
- Ground state for  $H$  will have form:  
$$\cdots |\phi_l\rangle \otimes |\phi_r\rangle \otimes |\phi_s\rangle \cdots$$

# H<sub>size</sub>

- $H_{size} = (1/n)I$

- $| \text{> < >} |$

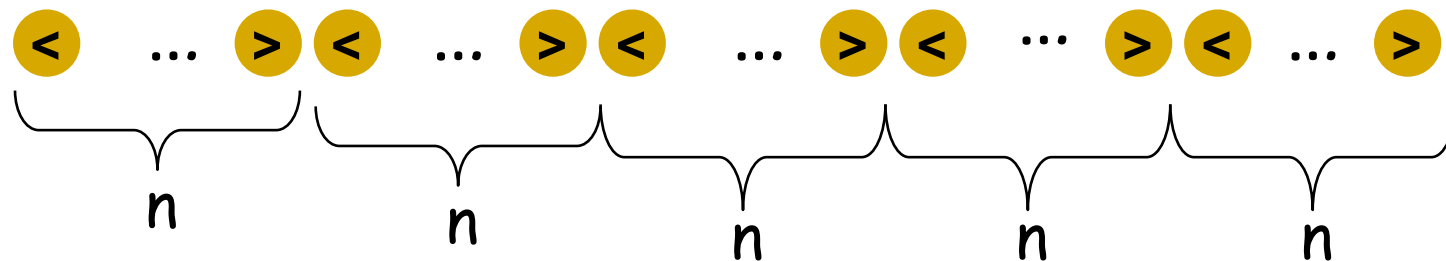
- +  $(n-1)/T_n [ | \uparrow \uparrow \times \uparrow \uparrow | + | \downarrow \downarrow \times \downarrow \downarrow | + | \uparrow \uparrow \times \uparrow \uparrow | ]$

- $T_l$  is the number of standard basis states in the support of the ground state for a segment of length  $l$ .

- $\langle \phi_l | H_{size} | \phi_l \rangle = 0$  if and only if  $l=n$

- Otherwise  $\langle \phi_l | H_{size} | \phi_l \rangle > \frac{1}{n^2}$

# Ground States for the Finite Cycle



- $n$  orthogonal ground states, each a translation one site over.
- For any region size a constant fraction of the regions of that size have high entanglement.
- Superposition of all  $n$  states is translationally invariant and for every region size, all regions of that size have high entanglement.



# Open Problems

- Improve gap - lower bound on entropy as a function of  $1/D$ .
- For a given region size  $m$ , can we achieve high entropy for all regions of size  $m$ ?
- Unique ground state for finite cycle and infinite chain?
- Can we achieve high entanglement entropy for all region sizes simultaneously for a translationally-invariant Hamiltonian?