# Irreversibility for all bound entangled states


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**[PRL 95, 190501 (2005),** ***quant/ph- 0506138*****]**

**RESQ (2003-2005), QUPRODIS (2003-2005), PBZ-MIN (2003)**

# Entangled states

## Mathematical definition:

**$\rho_{AB}$ is entangled** $\Longleftrightarrow$ **it cannot be decomposed as the** $\sum_i p_i \rho_i^A \otimes \rho_i^B$ **separable form**

## Operational asymptotic definition:

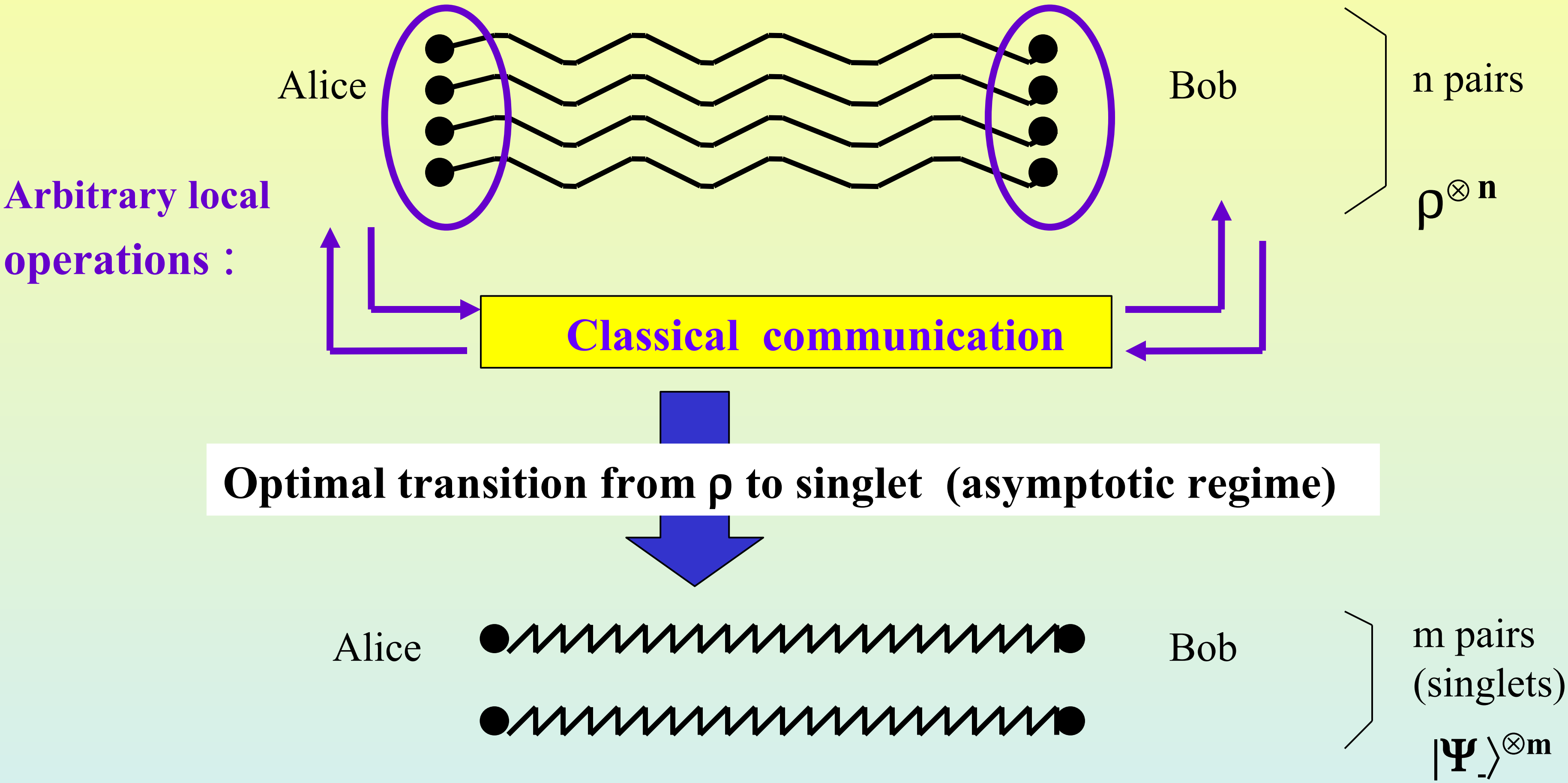
**$\rho_{AB}$ is entangled** $\Longleftrightarrow$ **it cannot be prepared by Local Operation and Classical Communication (LOCC) in asymptotic regime**

## Maximally entangled state (singlet)

$$|\psi_-\rangle = \frac{1}{\sqrt{2}}(|01\rangle - |10\rangle)$$

# Distillation of noisy entanglement

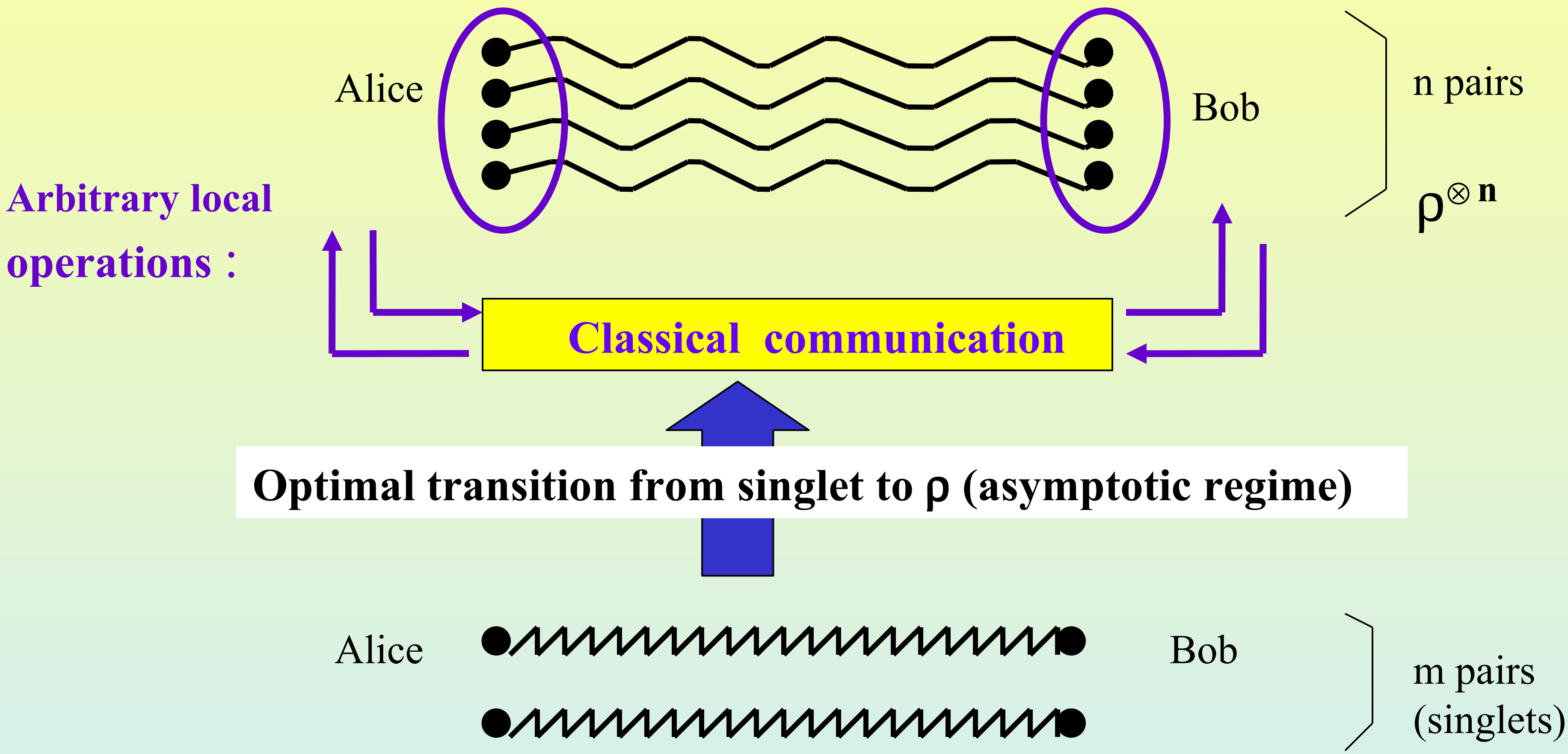
[Bennett,Brassard,Popescu, Schumacher, Smolin,Wooters PRL **76** 722-725 (1996) ]




Distillable entanglement: optimal rate $\mathbf{E_D(\rho) = m/n}$
(of singlets)

# Formation of entanglement

[Bennett, DiVincenzo, Smolin,Wootters, PRA **54**, 3824-3851 (1996)]




$|\Psi_-\rangle^{\otimes m}$

Entanglement cost : optimal rate $E_C(\rho) = m/n$ (of pairs)

# Reversibility for pure and separable states

## Separable (disentangled) states

1) No singlets  needed to create state: $E_C = 0$

3) No singlets can be drawn from state: $E_D = 0$

## Pure entangled states

$$\rho_{AB}^{\otimes n} \longrightarrow \psi_{+}^{\otimes m} \longrightarrow \rho_{AB}^{\otimes n}$$

$$E_D(\rho_{AB}) = E_C(\rho_{AB}) = S(\rho_A)$$

# Mixed states: irreversibility in entanglement theory

**Bound entangled states:**

[M.,P.,R. Horodeccy PRL **80** 5239-5242 (1998) ]

$\rho$ is entangled but $\rho^{\otimes n} \longrightarrow$ **No pure entangled states**

**Generic mixed state:**

[Rigorously: Vidal, Cirac PRL **86** 5803 (2001) ]

$\psi_+^{\otimes k} \longrightarrow \rho^{\otimes n} \longrightarrow \psi_+^{\otimes k'}$

$k' < k$

$E_D < E_C$

# Problem of irreversibility for bound entangled states

Are the processes of formation and distillation

**reversible** or **irreversible** ?

| | |
|---|---|
| $E_c(\rho_{AB}) = E_d(\rho_{AB})$ | **reversibility** |
| $E_c(\rho_{AB}) > E_d(\rho_{AB})$ | **irreversibility** |

# Entanglement needed to create one copy of state

**From bound entangled states, we cannot distill entanglement** $E_D = 0$ **but we need entanglement to create one copy of them, what refers nonzero value of** $E_F > 0$

## Entanglement of formation

$$E_F(\rho_{AB}) = \inf_{\{p_i,\varphi_i\}} \sum_i p_i S(|\varphi\rangle\langle\varphi|_A)$$

where $\rho = \sum_i p_i |\varphi_i\rangle\langle\varphi_i|$

# Problem of additivity for $E_F$

Relationship between $E_F$ and $E_C$

$$E_c(\rho) = \lim_{n\to\infty} E_F(\rho^{\otimes n})/n$$

[Hayden, M.Horodecki, Terhal, J.Phys.A, Math.Gen.2001]

**If $E_F$ is additive then $E_C = E_F$ and for all entangled states $E_C = E_F > 0$**

**Problem**

**We do not know if $E_F$ is additive**

# Other approaches to solve problem of irreversibility:

- **To find a new entanglement measure:**

$$E_C > E? > E_D$$

- **To find lower bound of $E_C$:**

$$E_C \geq G \;\text{i}\; G > 0$$

**for bound entangled states**

# Towards construction of G

Lets recall a measure of classical correlation of bipartite state:

[Henderson, Vedral JPA **34**, 6899 (2001)]

$$C(\rho_{AB}) = \max_{\{BB^{+}\}} S(\rho_A) - \sum_i p_i S(\rho_A^i)$$

where

$$\rho_A^i = \frac{1}{p_i} Tr_B(I_A \otimes B_i \rho_{AB} I_A \otimes B_i^{+})$$

$$p_i = Tr(I_A \otimes B_i)\rho_{AB}(I_A \otimes B_i^{+})$$

# Properties of C

$$C(\rho_{AB}) = 0 \text{ iff } \rho_{AB} = \rho_A \otimes \rho_B$$

**C is invariant under local unitary operations**

**C is non-increasing under local operations, in particular**

$$C(\rho_{AA':B}) \geq C(\rho_{A:B})$$

# Duality relation

[M. Koashi and A.Winter, PRA **69**, 022309(2004)]

For a tripartite pure state $\varphi_{ABC}$, the following duality relation is satisfied:

$$S(\rho_A) = E_F(\rho_{AB}) + C(\rho_{AC})$$

where $\rho_{AB} = Tr_C \varphi_{ABC}$, is dual to $\rho_{AC} = Tr_B \varphi_{ABC}$ and vice versa.

**Note:**

$$S(\rho_A) = \underbrace{\min_{\{p_i,\rho_i\}} \sum_i p_i S(\rho_A^i)}_{\mathbf{E_F}} + \underbrace{\left(S(\rho_A) - \min_{\{p_i,\rho_i\}} \sum_i p_i S(\rho_A^i)\right)}_{\mathbf{C}}$$

# Definition a candidate for bound on $E_C$

Lets define some new quantities for $\rho_{AB}$

$$G(\rho_{AB}) = \inf_{\{p_i, \rho_i\}} \sum_i p_i C(\rho_i)$$

where infimum is taken over all in general mixed ensemble realizing state $\rho_{A:B} = \sum_i p_i \rho^i_{A:B}$

# G is positive for all entangled state

## Theorem

$$G(\rho_{A:B}) = 0 \Leftrightarrow \rho_{A:B} \text{ is separable state}$$

**Proof.**

$"\Rightarrow"$

If G is equal to zero then C must be zero for every element of ensemble $\{p_i, \rho^i_{AB}\}$ realizing ρ, so all states in ensemble must be product $\{\rho_i^{AB} = \rho_i^A \otimes \rho_i^B\}$

# For all separable state G=0

- $C(\rho_{A:B})$ **is continuous**

this comes from duality relation but it is also a consequence of proposition proved in our paper about asymptotic continuity:

[see: B.Synak-Radtke, M.Horodecki, quant-ph/0507126 ]

- Basing on Caratheodory theorem we can show that for G, being convex roof of other continuous function, **there exists optimal decomposition** $\{p_i, \rho^i_{AB}\}$
- If the state is entangled, there must be a **non-product state in decomposition** and C in nonzero on this state.

⇓

**G > 0 for entangled states and G = 0 for separable states**

# Inequality relating C and $E_F$

## Lemma

For any four-partite pure state $\psi_{AA'BB'}$ the following inequality of entanglement is satisfied:

$$E_F(\psi_{AA':BB'}) \geq E_F(\rho_{A:B}) + C(\rho_{A':B'})$$

where $\rho_{AB} = Tr_{A'B'}\psi_{AA'BB'}$ and $\rho_{A'B'} = Tr_{AB}\psi_{AA'BB'}$

**Proof.** We apply a duality relation to a 4-partite state $\psi_{AA'|B|B'}$

$$S(\rho_{AA'}) = E_F(\rho_{\mathrm{AA':B}}) + C(\rho_{\mathrm{AA':B'}}) \geq$$

$$\geq E_F(\rho_{\mathrm{A:B}}) + C(\rho_{\mathrm{A':B'}})$$

# Inequality relating G and $E_F$

**Proposition**

For a mixed four-partite state $\rho_{A'ABB'}$

$$E_F(\rho_{\mathrm{AA':BB'}}) \geq E_F(\rho_{\mathrm{A:B}}) + G(\rho_{\mathrm{A':B'}})$$

**Proof.** Let $\{p_i, \varphi^i_{AA'BB'}\}$ be the optimal realization of $E_F$ of state $\rho_{AA':BB'}$ then

$$E_F(\rho_{\mathrm{AA':BB'}}) = \sum_i p_i S(\rho^i_{AA'}) \geq \sum_i p_i E_F(\rho^i_{\mathrm{A:B}}) + \sum_i p_i C(\rho^i_{\mathrm{A':B'}})$$

$$\geq E_F(\rho_{\mathrm{A:B}}) + \sum_i p_i C(\rho^i_{\mathrm{A':B'}}) \geq E_F(\rho_{\mathrm{A:B}}) + G(\rho_{\mathrm{A':B'}})$$

# G is lower bound of $E_C$

**Theorem**

For any entangled state $\rho_{AB}$

$$E_C(\rho_{\mathrm{A:B}}) \geq G(\rho_{\mathrm{A:B}}) > 0$$

**Proof.** Notice that $E_c = \lim_{n\to\infty} E_F(\rho^{\otimes n})/n$, then

$$E_F(\rho^{\otimes n}) = E_F(\rho^{\otimes n-1} \otimes \rho) \geq E_F(\rho^{\otimes n-1}) + G(\rho) \geq \ldots$$

$$\geq \ldots \geq E_F(\rho) + (n-1)G(\rho)$$

So $$\frac{E_F(\rho^{\otimes n})}{n} \geq \frac{n-1}{n} G(\rho)$$

Let $n \to \infty$ then $E_C(\rho_{A:B}) \geq G(\rho_{A:B}) > 0$

# Irreversibility for all bound entangled state

For any entangled state $\rho_{AB}$

$$E_C(\rho_{A:B}) > 0$$

For all bound entangled states $\mathbf{E_D = 0}$**, but** $\mathbf{E_C > 0}$

**Irreversibility between process of formation and distillation for all bound entangles states!!!!**

D. Yang, M. Horodecki, R .Horodecki and B. Synak-Radtke „Irreversibility for all bound entangled states", PRL **95**, 190501 (2005), *quant/ph- 0506138,*

# Application 1

$$E_C(\rho_{\text{entangled}}) > 0$$

**Mathematical definition of entangled states is equivalent to operational asymptotic one!!!**

# Application 2

For a four-partite state $\rho_{AA'BB'}$

$$E_F(\rho_{AA':BB'}) \geq E_F(\rho_{A:B}) + G(\rho_{A':B'})$$

if the reduced state is $\rho_{A'B'}$ entangled then

$$E_F(\rho_{AA':BB'}) > E_F(\rho_{A:B})$$

⇓

**It is impossible to clone a known entangled state by LOCC.**

[**See:** M. Horodecki, A.Sen (De), U. Sen, Phys.Rev.A **70** 052326 (2004)]

**Thank you**

**for your attention !**