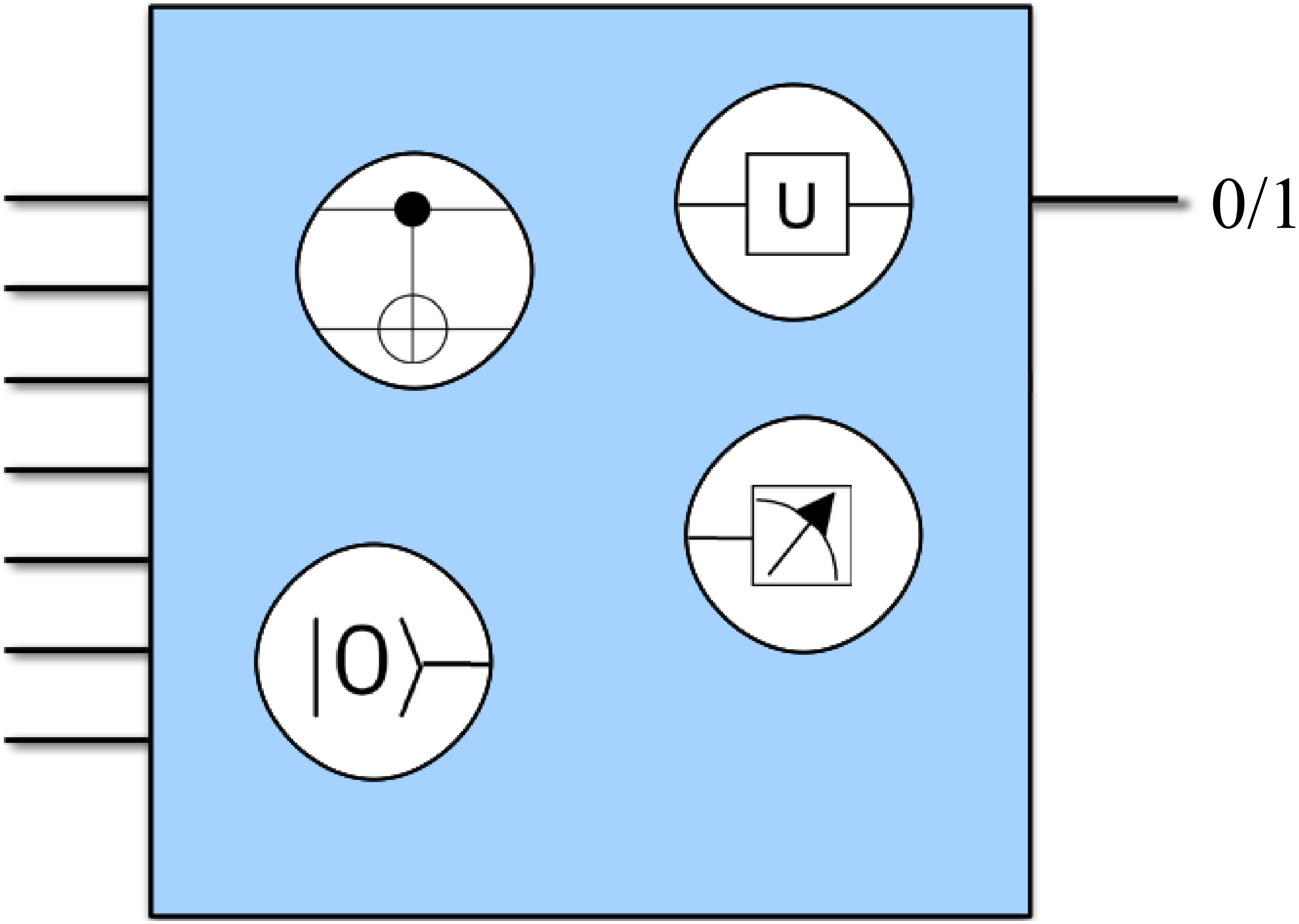
Rigorous fault-tolerance thresholds

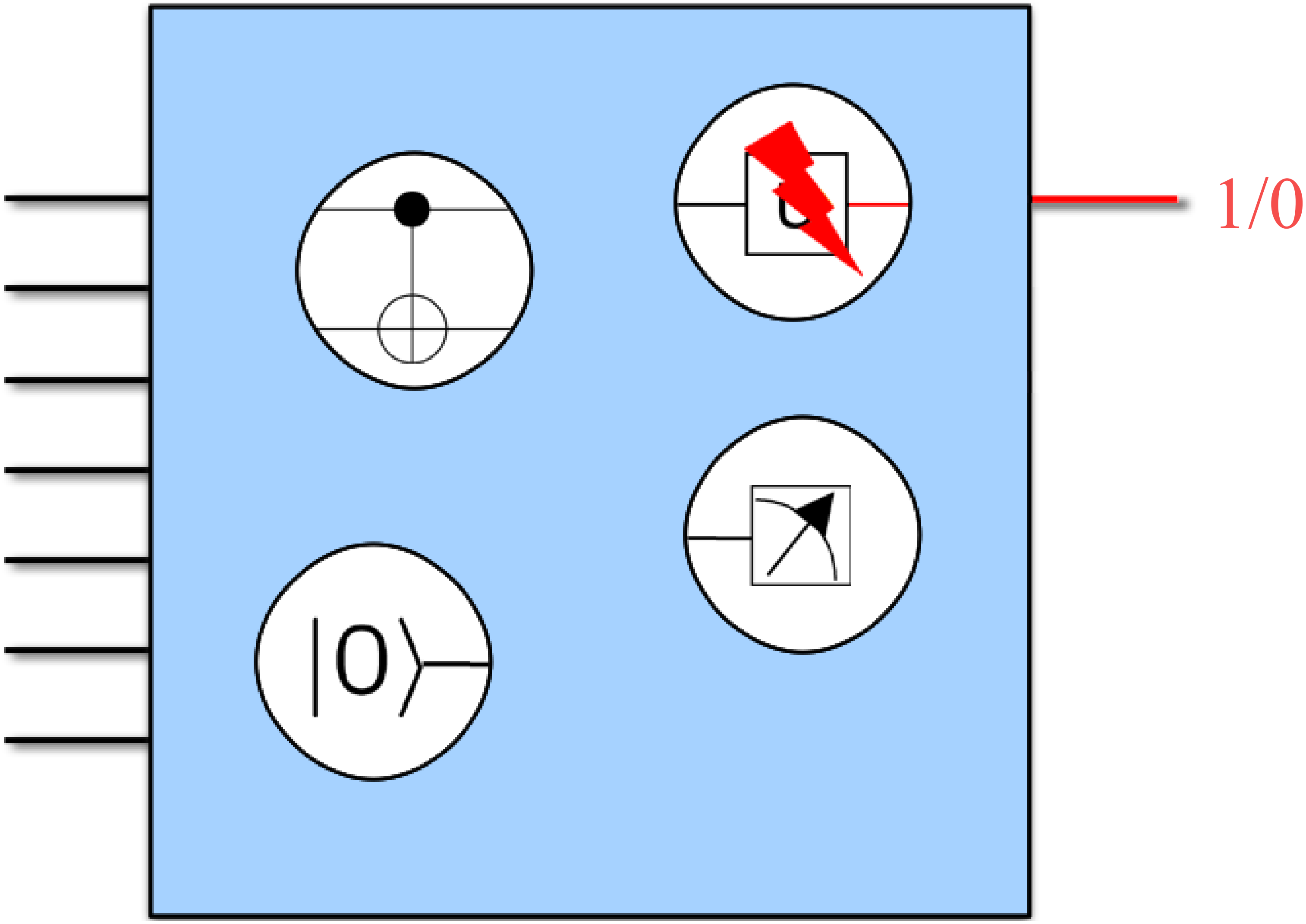
Ben Reichardt

UC Berkeley

N gate circuit

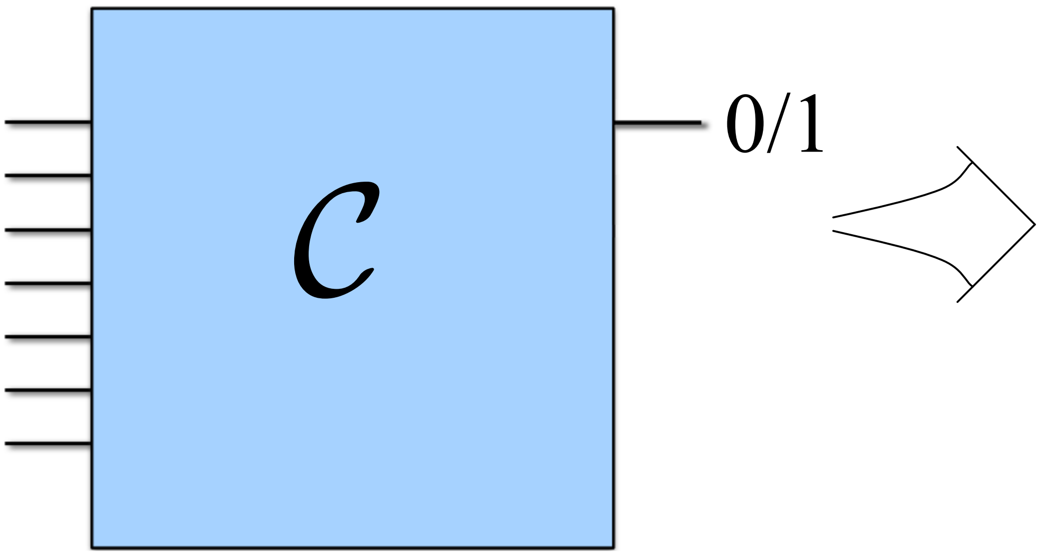


N gate circuit ⇒ Need error $\ll 1/N$



Quantum fault-tolerance problem

– Classical fault-tolerance: Von Neumann (1956)



Fault-tolerant, larger $\mathcal{C}$

- High tolerable noise
- Low overhead

Quantum fault-tolerance problem

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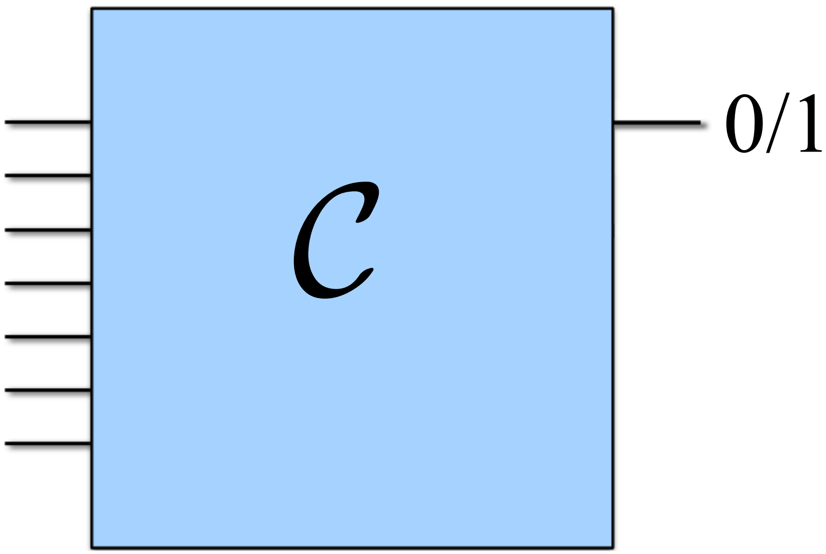
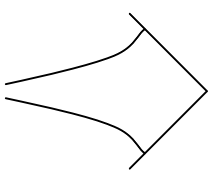


Important problem!



Fault-tolerant, larger

$\mathcal{C}$

- High tolerable noise
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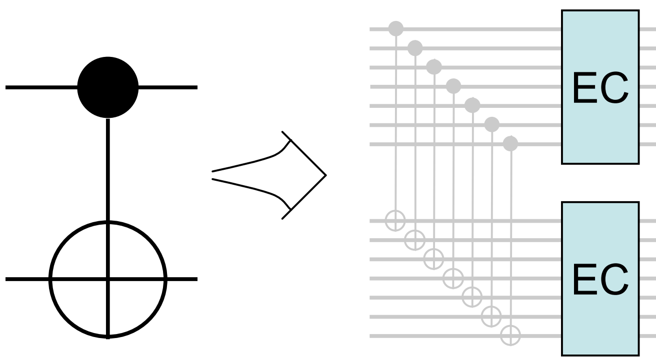
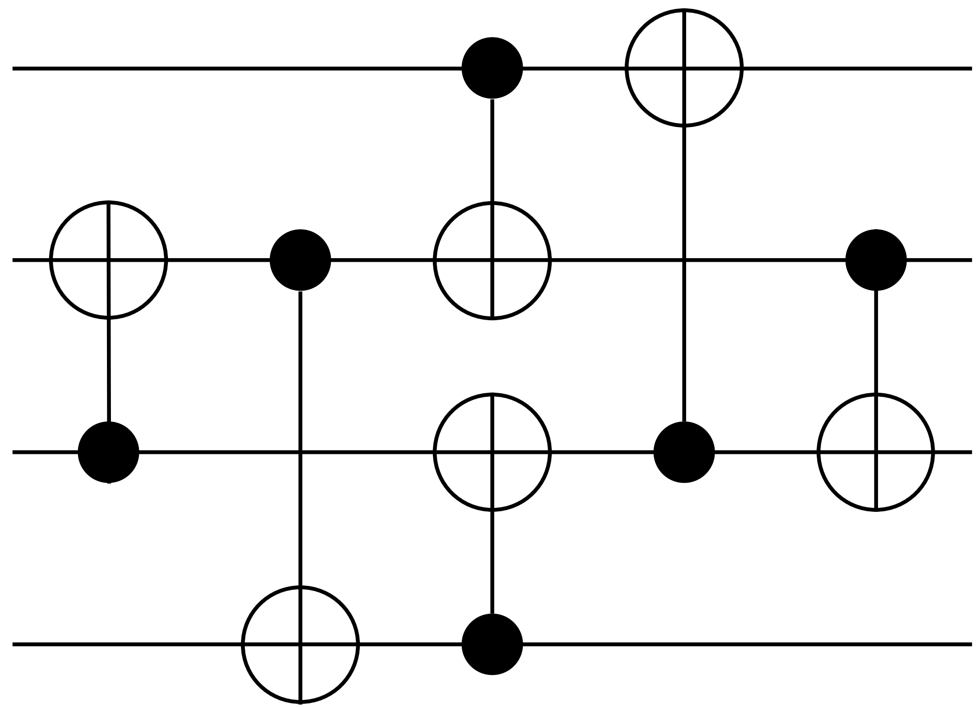
Intuition

- Work on encoded data
- Correct errors to prevent spread
- Concatenate procedure for arbitrary reliability





- Quantum fault-tolerance: Shor (1996)
  - Using poly(log N)-sized code, tolerate 1/poly(log N) error
- Aharonov & Ben-Or ('97), Kitaev ('97), Knill-Laflamme-Zurek ('97)
  - Using concatenated constant-sized code, tolerate constant error

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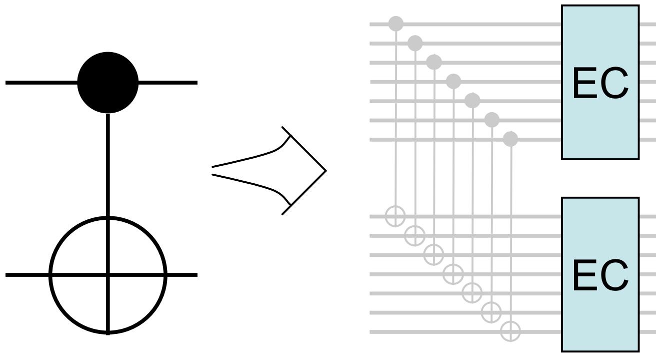

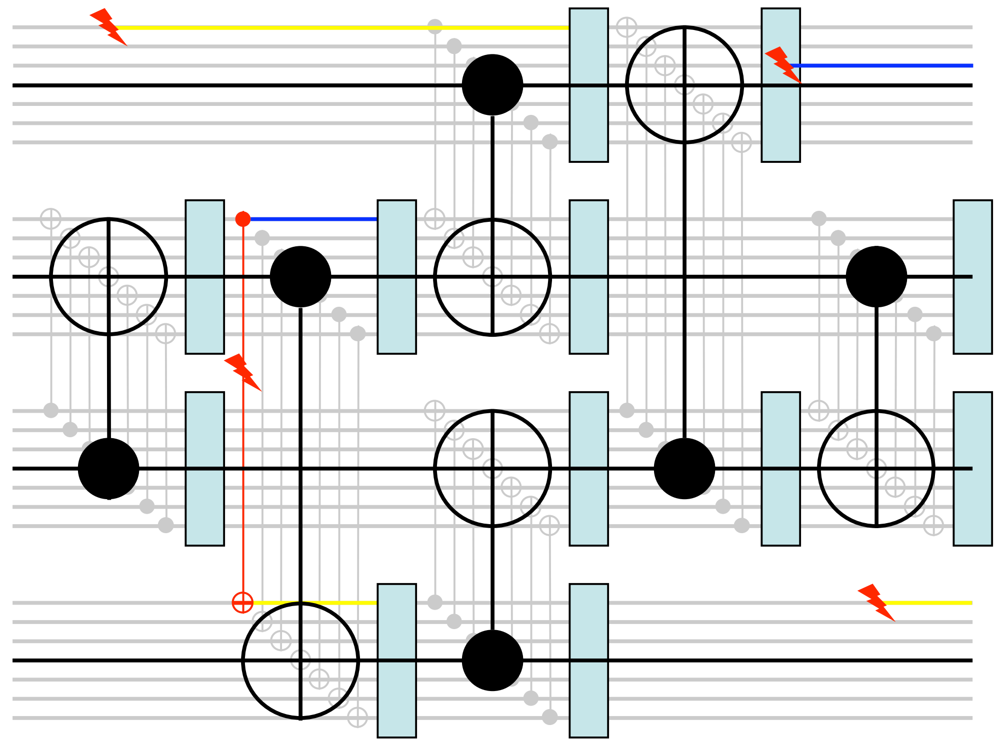

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  - – Using concatenated constant-sized code, tolerate constant error

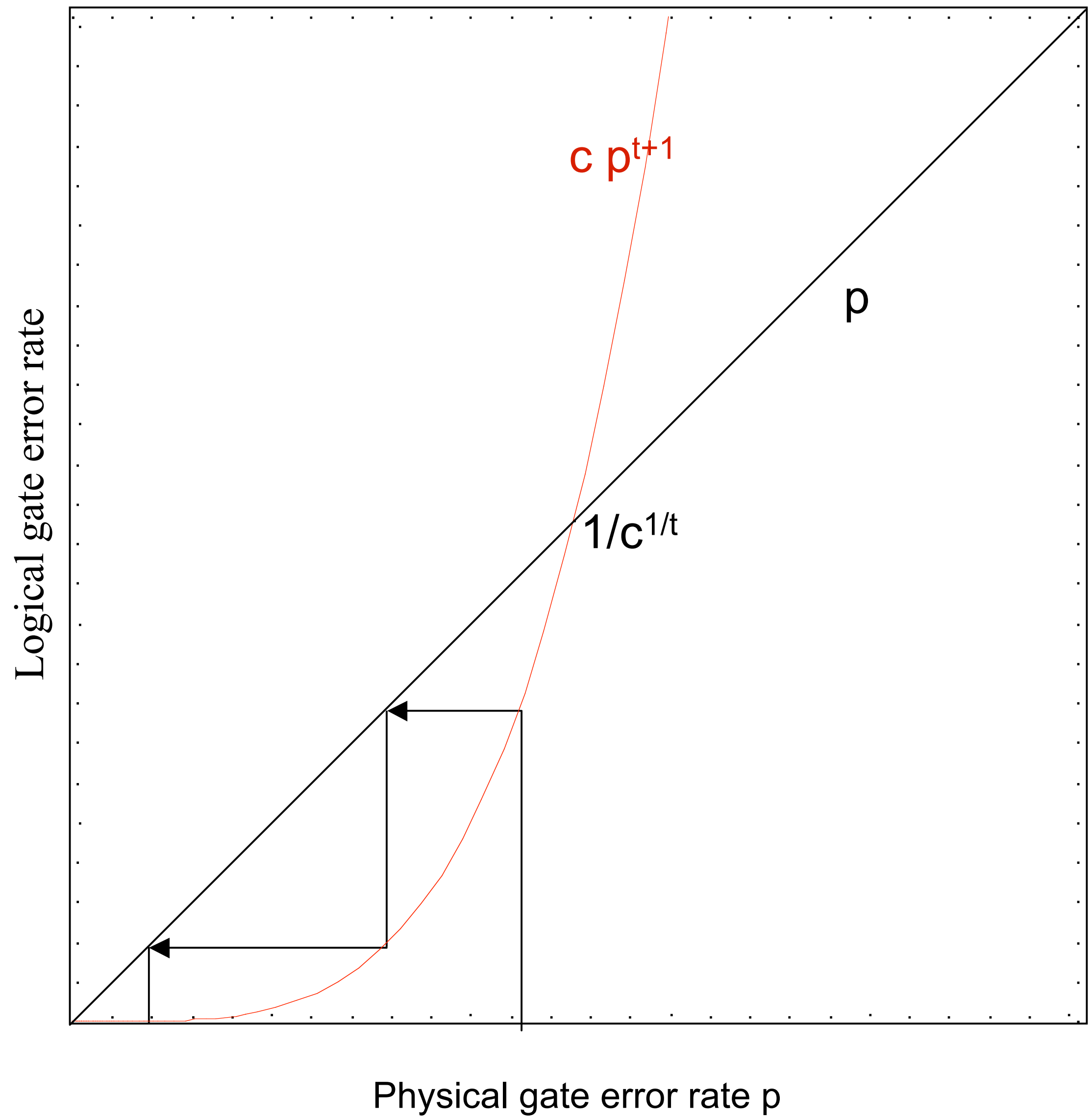
Concatenation

- N gate circuit
  - $\Rightarrow$ Want error $\ll 1/N$

- m-qubit, t-error correcting code

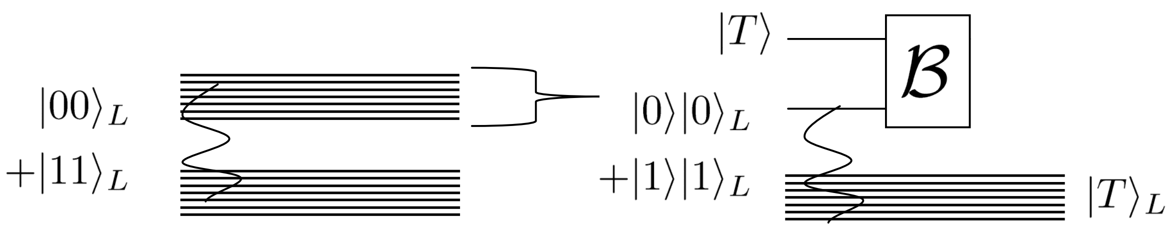
Probability of error	Physical bits per logical bit
p	1
$c\,p^{t+1}$	m
$\sim p^{(t+1)^2}$	m^2
$p^{(t+1)^3}$	m^3

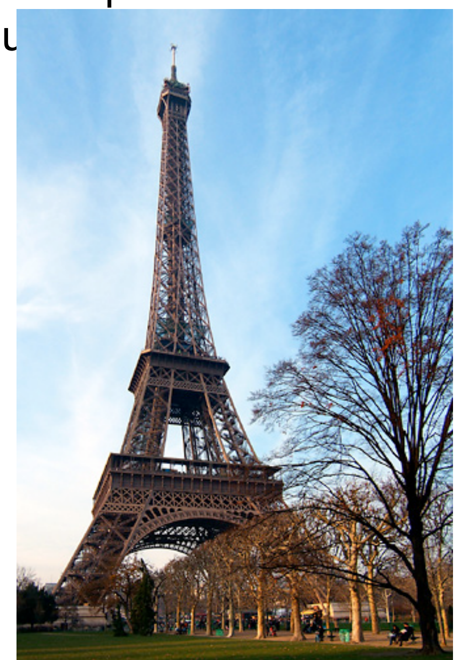
O(log log N) concatenations
poly(log N) physical bits / logical

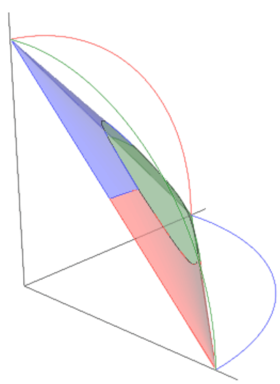


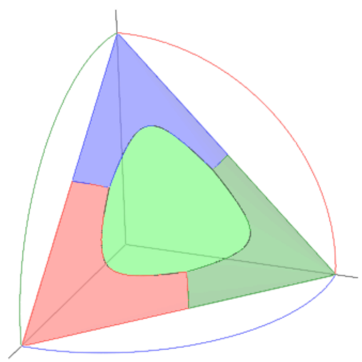
Recent results

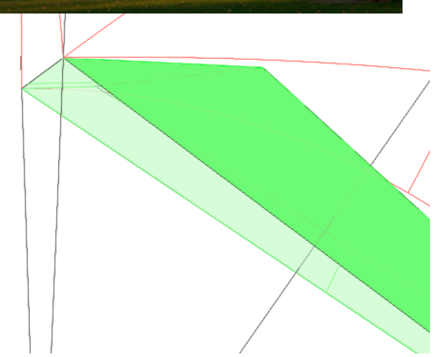
- Magic states distillation [Bravyi & Kitaev '04, Knill '04]
  - Universality method, related to best current threshold upper bounds
  - Reduction Universal fault-tolerance ⇝ Stabilizer op. fau











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  - Erasure error threshold is 1/2 for Bell measurements
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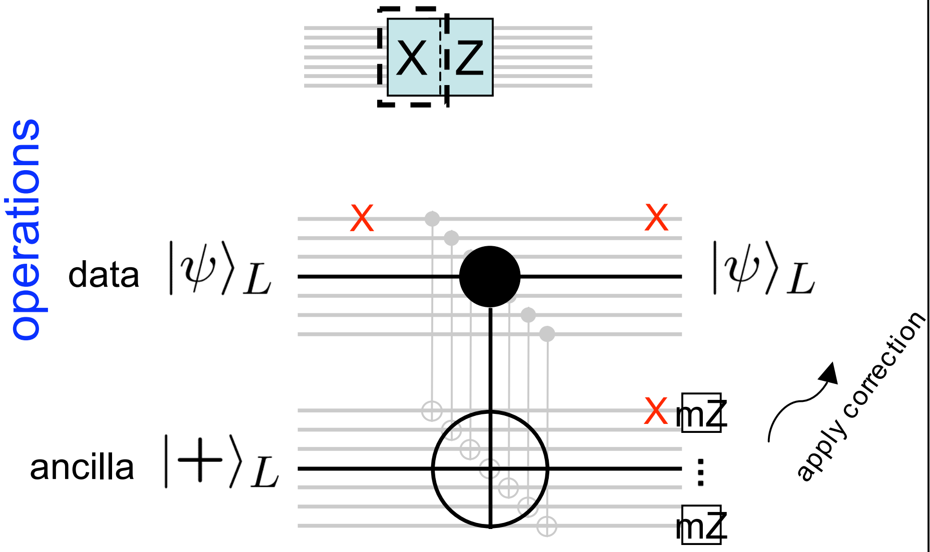
    1% with substantial but more reasonable overhead

Fault-tolerance threshold myth:

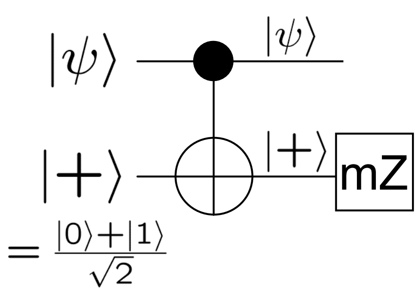
Threshold is all that counts.
Maximize the threshold at all costs.

Steane-type error correction

Physical operations



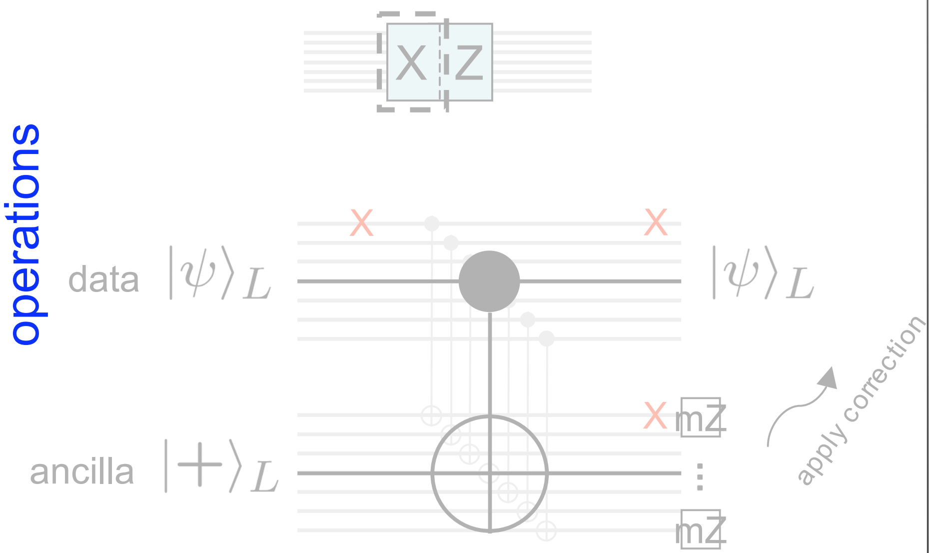
Logical operations

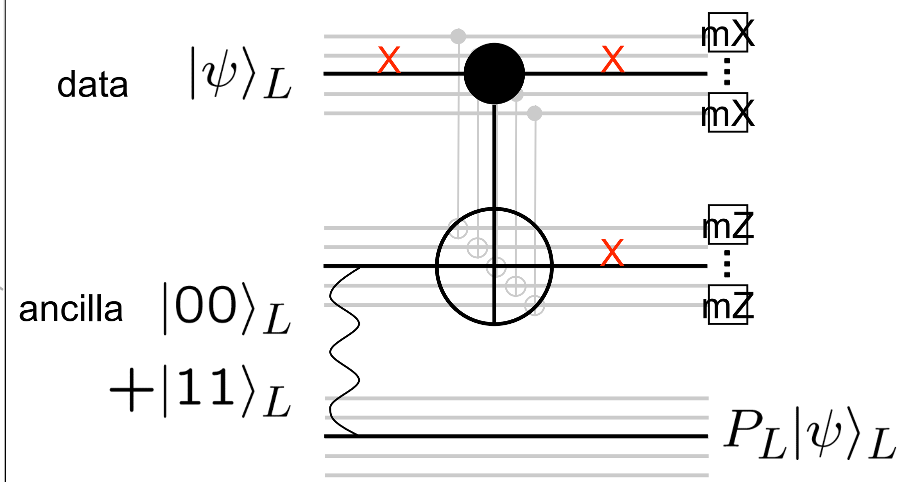


Steane-type error correction

Knill-type error correction

Physical operations





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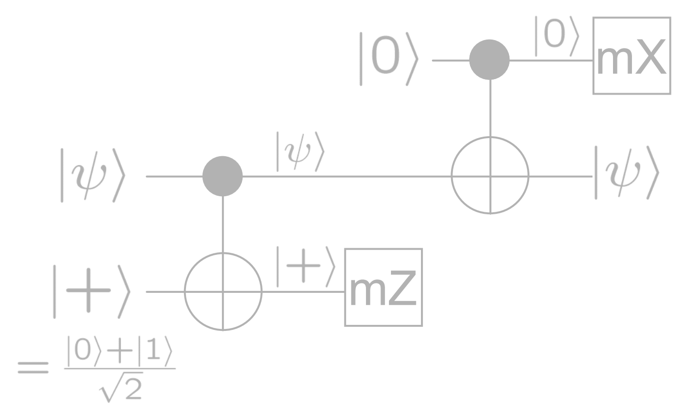
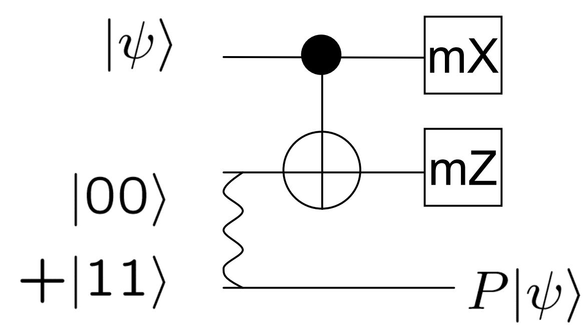
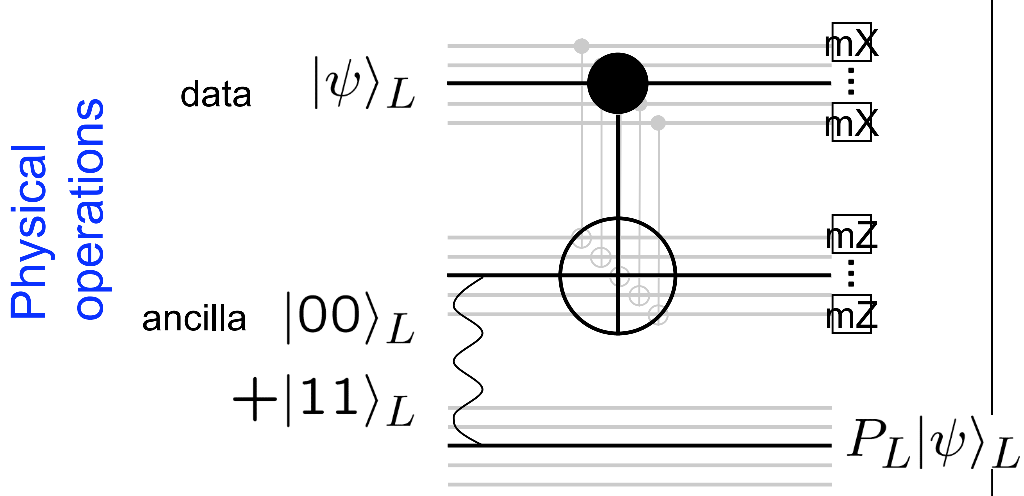


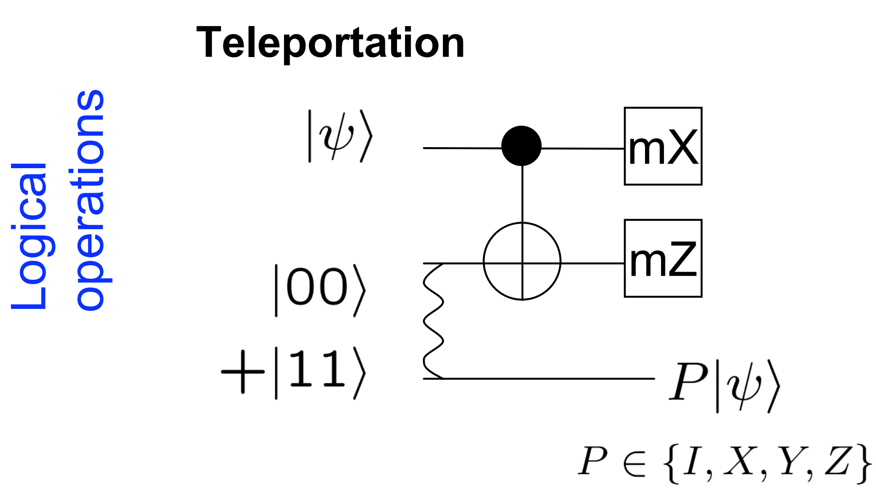
Teleportation



$P \in \{I, X, Y, Z\}$

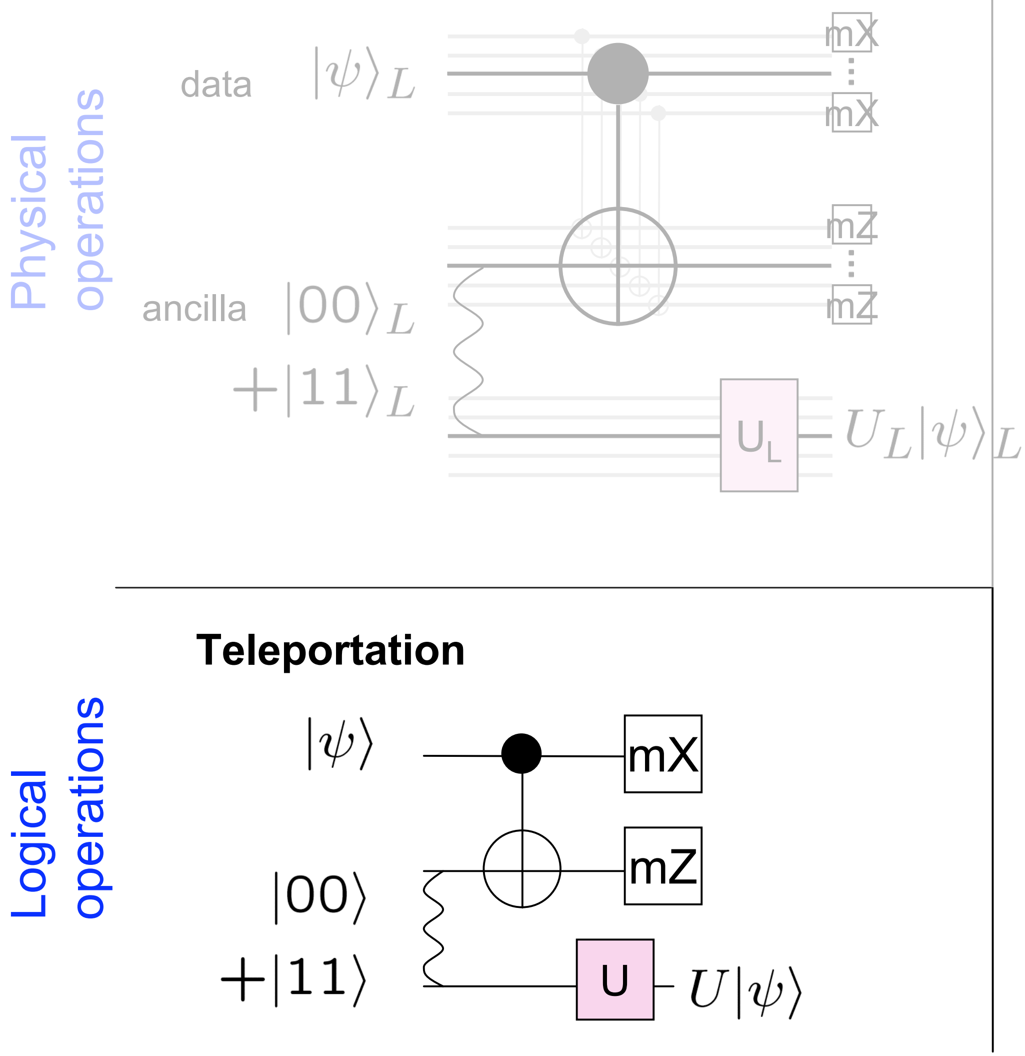
Knill-type error correction





- Advantages
  - Efficient
  - Technical advantage: Reduces blockwise independence to encoded Bell state

Knill-type correction + computation



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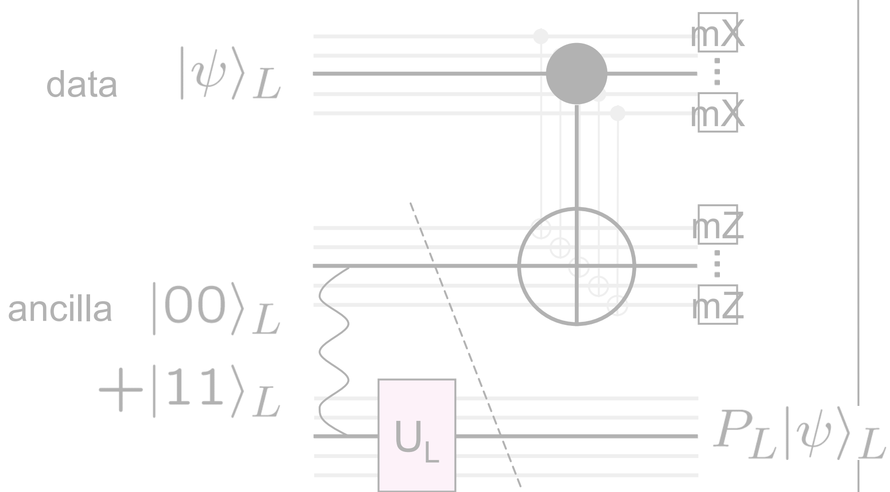

Logical operations

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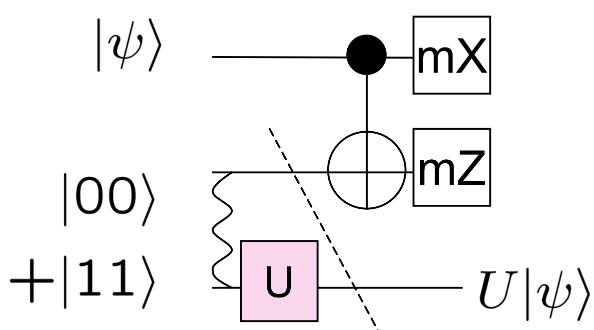

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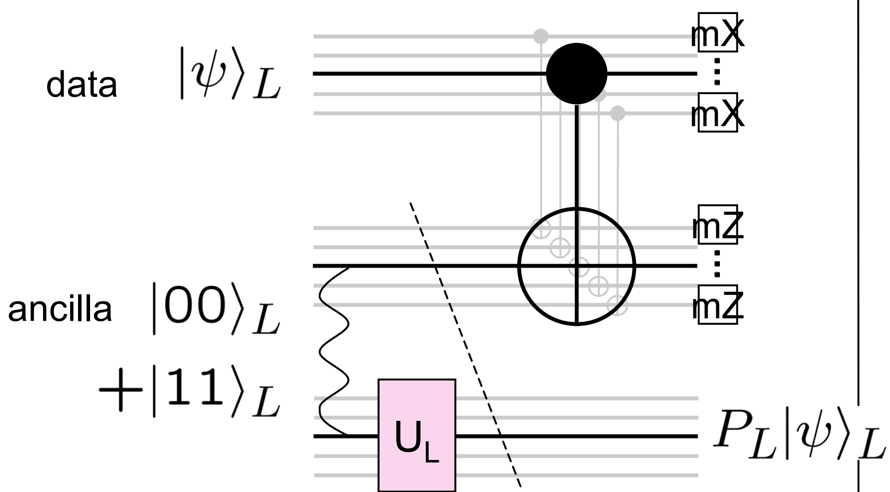
Knill-type correction + computation

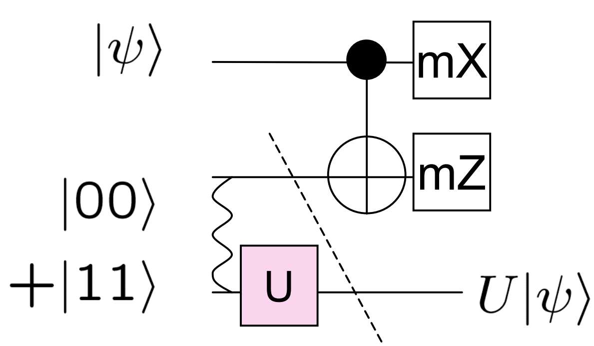
+ Distance-two code
+ Postselection

Physical operations



Logical operations

Teleportation



- Advantages
  - Efficient
  - Technical advantage: Reduces blockwise independence to encoded Bell state
  - Allows for more checking
- Disadvantages
  - High overhead at high error rates with error detection
  - Renormalization penalty requires stronger control over error distribution
  - No threshold has been proved to exist

Main issues

- Bounded dependencies
  - Between different blocks
  - In time
  - Between bit errors and logical errors
- Example:

$|0\rangle_L$ w/ prob. 1-q $|1\rangle_L$ w/ prob. q

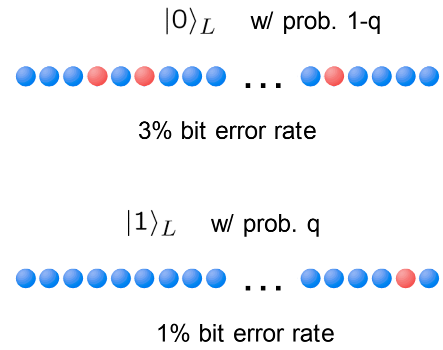
3% bit error rate   1% bit error rate

accepted w/ prob. $(1\text{-}q)\ .97^n$ $q\ .99^n$

$\Rightarrow$ Probability of logical error increases exponentially!

Main issues

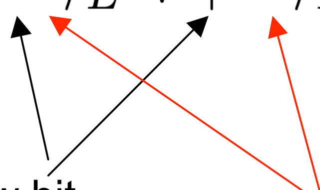
- Bounded dependencies
  - Between bit & logical errors



Monotonicity?

want encoded Bell pair: $|00\rangle_L + |11\rangle_L$

get: $|01\rangle_L + |10\rangle_L$

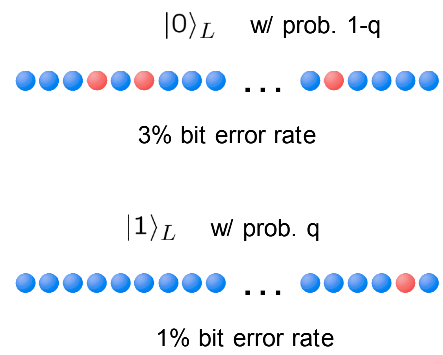


monotonicity $\Rightarrow$ low bit error rate, high bit error rate

But! $|01\rangle_L + |10\rangle_L$

Main issues

- Bounded dependencies
  - Between bit & logical errors



Monotonicity?

(repetition code)

$$|0\rangle_{L(k)}$$

$$|0\rangle_{L(k-1)}\ |0\rangle_{L(k-1)}\ |{\color{red}1}\rangle_{L(k-1)}$$

$$|0\rangle_{L(k-2)}\ |{\color{red}1}\rangle_{L(k-2)}\ |{\color{red}1}\rangle_{L(k-2)}$$

Recent results (continued)

- Magic states distillation [Bravyi & Kitaev '04, Knill '04]
  - Universality method, related to best current threshold upper bounds
  - Reduction from FT universality to FT stabilizer operations
- Optimized fault-tolerance schemes: [Knill '03]
  - Erasure error threshold is 1/2 for Bell measurements
  - [Knill '05]: > ~~5%~~ estimated threshold for depolarizing noise
    1% with substantial but more reasonable overhead
- Improved threshold *proofs*
  - Aliferis/Gottesman/Preskill '05: 2.7×10^{-5} } more efficient
  - R. '05: $< 1.4 \times 10^{-5}$ } distance three
  - Ouyang, R. (unpublished): 10^{-4}

Distance-3 code thresholds

- Basic estimates
  - Aharonov & Ben-Or (1997)
  - Knill-Laflamme-Zurek (1998)
  - Preskill (1998)
  - Gottesman (1997)
- Optimized estimates
  - Zalka (1997)
  - R. (2004)
  - Svore-Cross-Chuang-Aho (2005)
- 2-dimensional locality constraint
  - Szkopek et al (2004)
  - Svore-Terhal-DiVincenzo (2005)

- But no constant threshold was even proven to exist for distance-3 codes!
  - Aharonov & Ben-Or proof only works for codes of distance at least 5
- Today: Threshold for distance-3 codes

Dist-2 code threshold & threshold gap

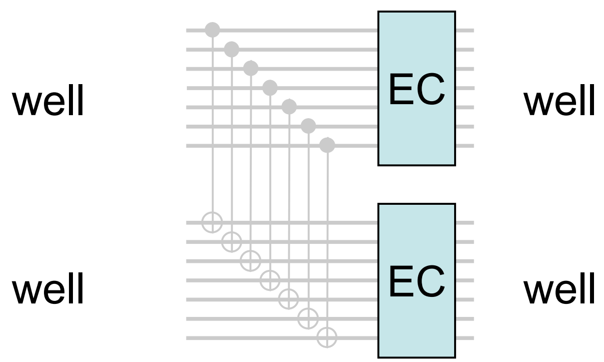
- Knill (2005) has highest threshold estimate ~5%
  - … Albeit with large constant overhead (more reasonable at 1%)
  - Again, no threshold has been proved to exist

- Gaps between proven and estimated thresholds
  - Estimates are as high as ~5%
  - Aliferis-Gottesman-Preskill (2005): 2.6×10^{-5}

- Caveat: Small codes aren't necessarily the most efficient
  - Steane ('03) found 23-qubit Golay code had higher threshold (based on simulations), particularly with slow measurements
  - 23-qubit Golay code proven: 10^{-4}

Distance-three code threshold proof intuition

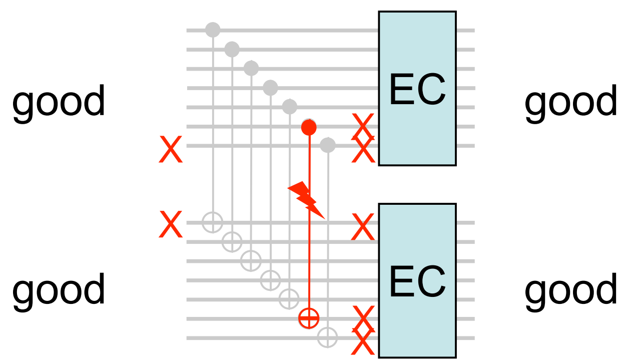
- **Idea:** Maintain inductive invariant of wellness. (A block is well "if it has at most one unwell subblock, and that only rarely.")



What's new: Control *probability distribution* of errors, not just error states.

Aharonov/Ben-Or-style proof intuition

- **Idea:** Maintain inductive invariant of goodness. (A block is good "if it has at most one bad subblock.")



(assuming one level k-1 error, m≥7)

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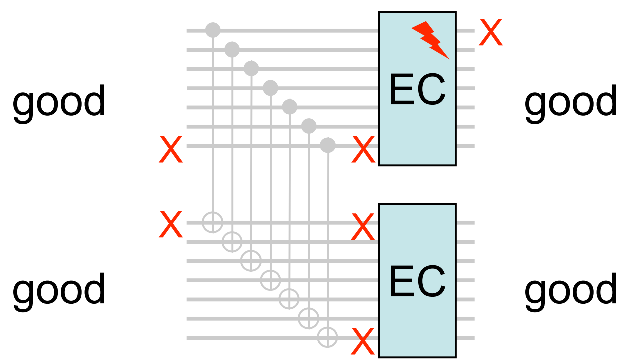

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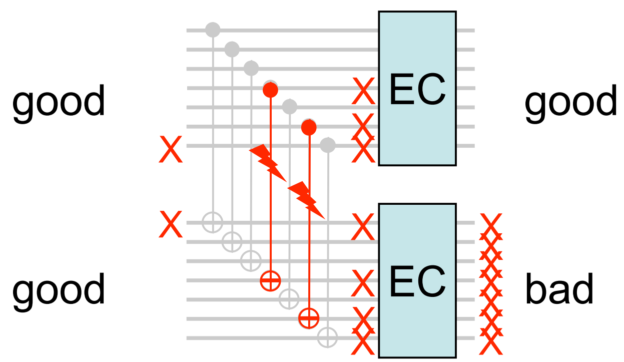

(two level k-1 errors, m=7)

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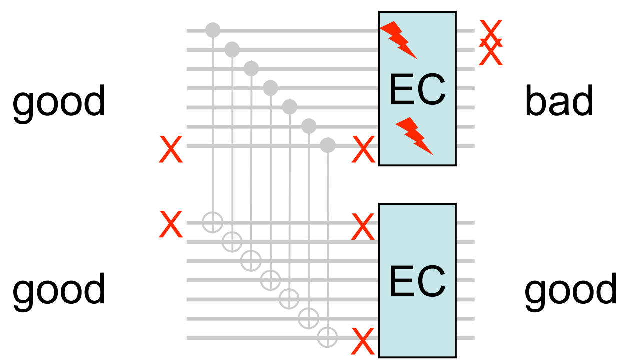
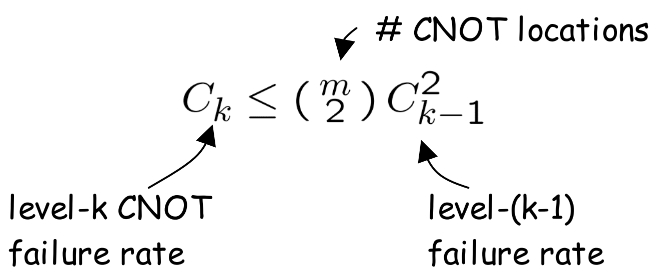


(two level k-1 errors)



$$C_k \leq \binom{m}{2} C_{k-1}^2$$

Aharonov/Ben-Or-style proof intuition

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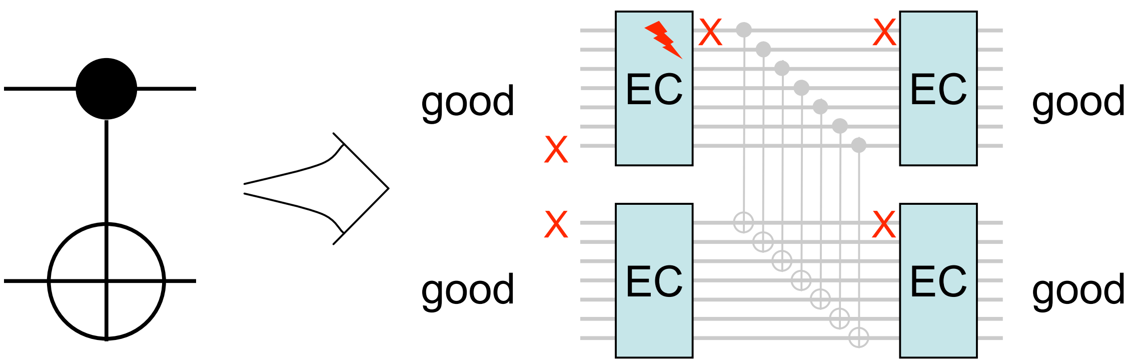
For distance-5 code:



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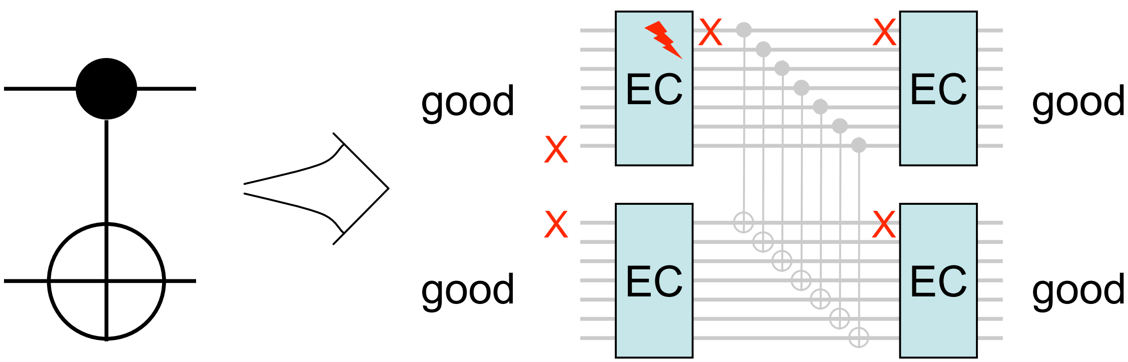
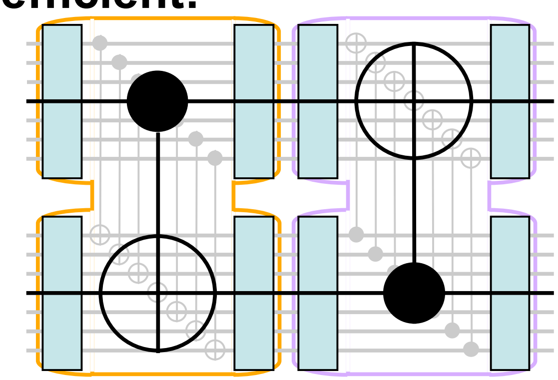


- **Inefficient:**

1.

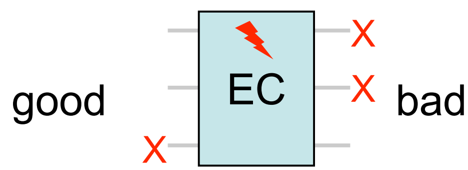


2. $p \to \binom{m}{2} p^2$ not cp^3 (distance = 5)

3. No threshold for concatenated distance-three codes.

Aharonov/Ben-Or-style proof intuition

- **Idea:** Maintain inductive invariant of goodness. (A block is good “if it has at most one bad subblock.”)
- Why not for distance-three codes?



(one level k-1 error is already too many)

- **New idea:** Most blocks should have no bad subblocks. Maintain inductive invariant of a controlled probability distribution of errors: “wellness.” (A block is well “if it only rarely has a bad subblock.”)

Proof overview

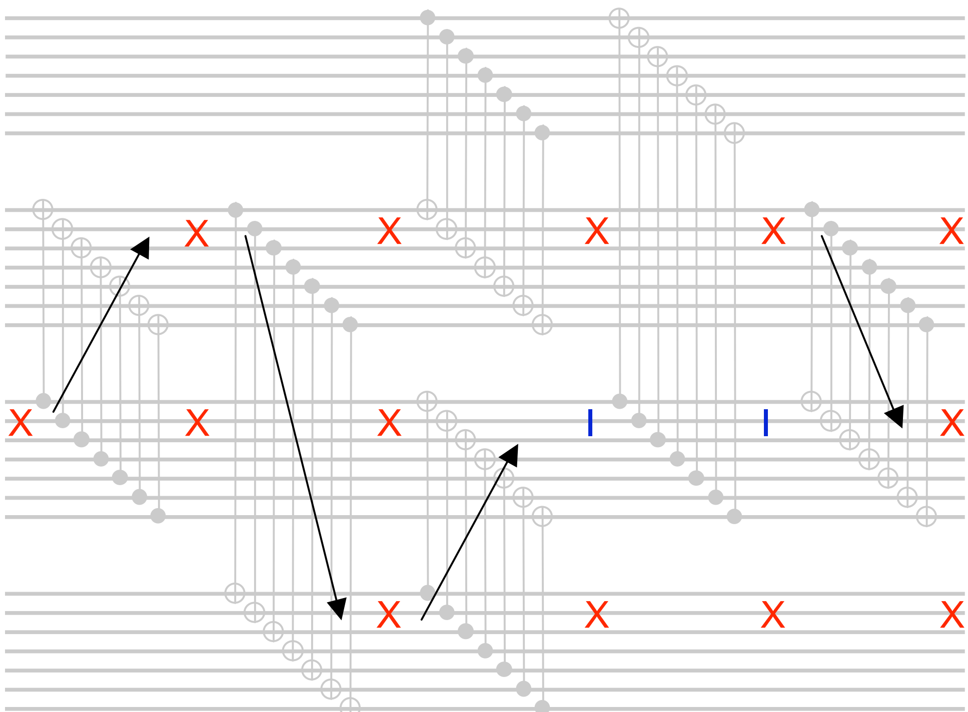
- Def: Error states (resolve $|01\rangle + |10\rangle$ ambiguity)
- Def: Relative error states (encoded CNOT must work even on erroneous input)
- **Def: good block**
- **Def: "well" block**
- Distance-3 code threshold setup
- Def: Logical success and failure
- Distance-3 code threshold proof

Def: Error states

- **Problem:** Different errors are equivalent, so it is ambiguous which bit is in error $|01\rangle + |10\rangle$
- **Solution:** Track errors from their introduction

Def: Error states

- **Tracking errors**



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$$|01\rangle + |10\rangle$$

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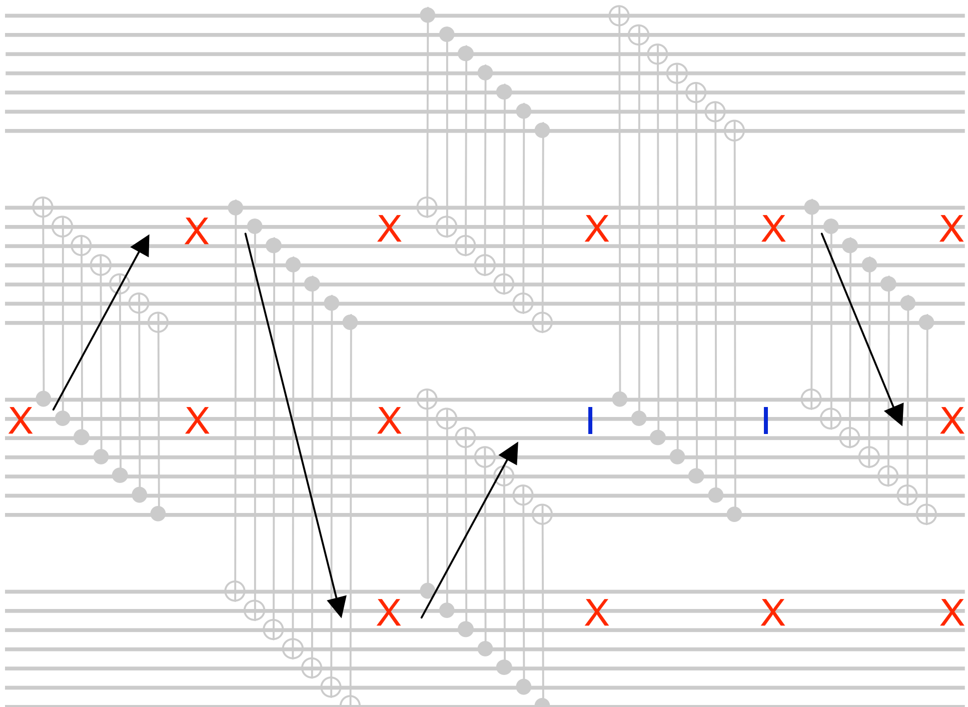
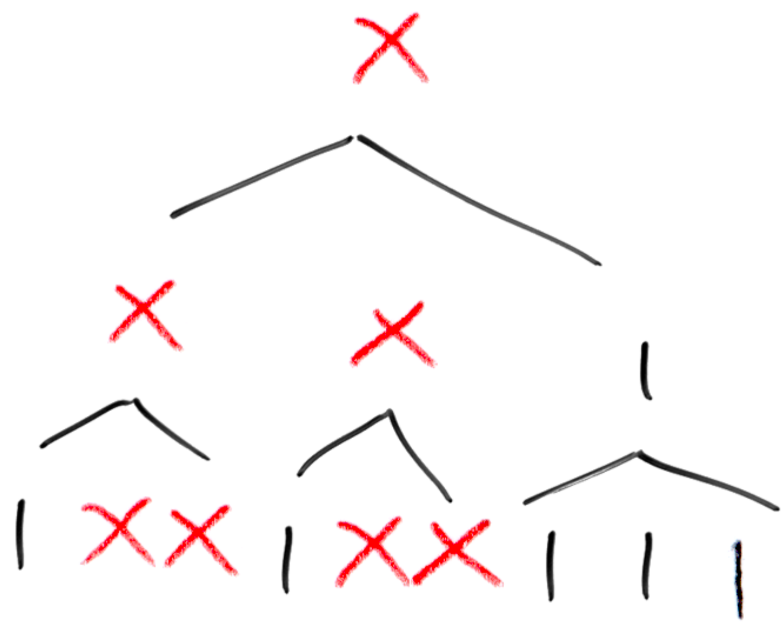
Def: Error states

- **Tracking errors**

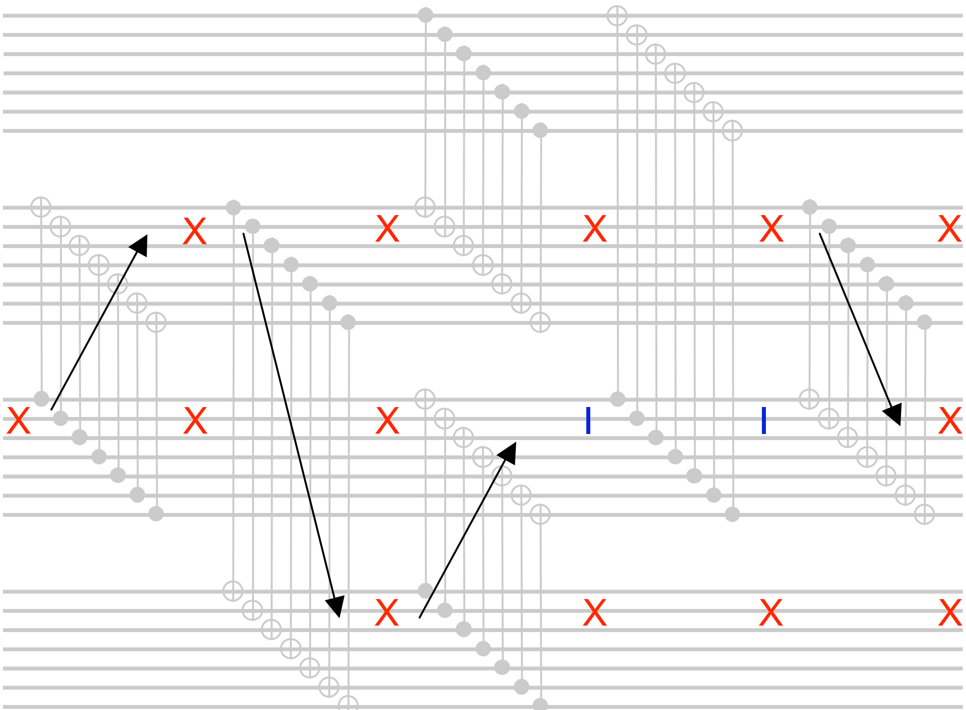


- **Block error states:** ideal recursive decoding



Def: *Relative* Error states

- **Tracking errors**



- **Block error states:** ideal recursive decoding
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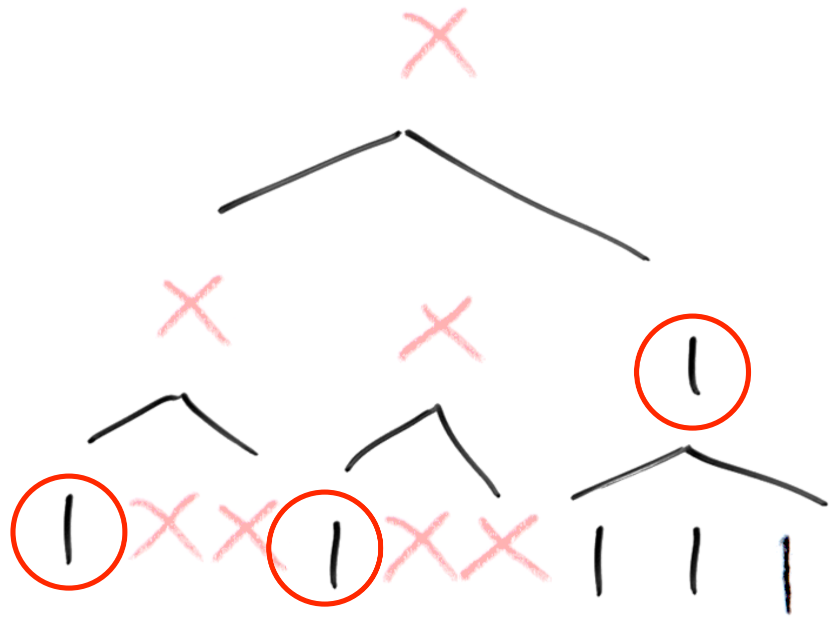
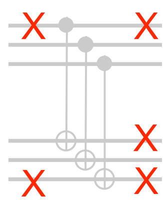


Def: good

- **Tracking errors**



- **Block error states:** ideal recursive decoding
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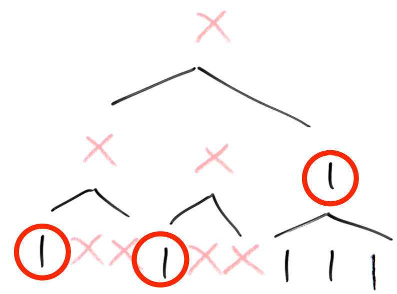
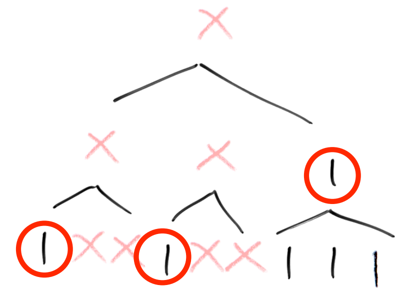


- **Def:** A block_k is good_k if it has at most one subblock_{k-1} either in relative error or not good_{k-1} itself.
(Every bit [$\equiv \text{block}_0$] is good_0.)

good examples

- **Relative error states** based on ideal recursive decoding
- A **good** block has at most one subblock either in relative error or bad.



good

bad

good examples

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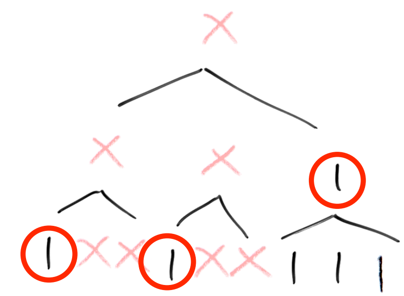
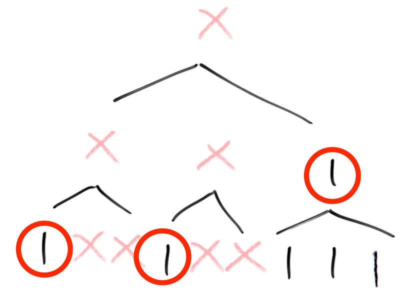


good

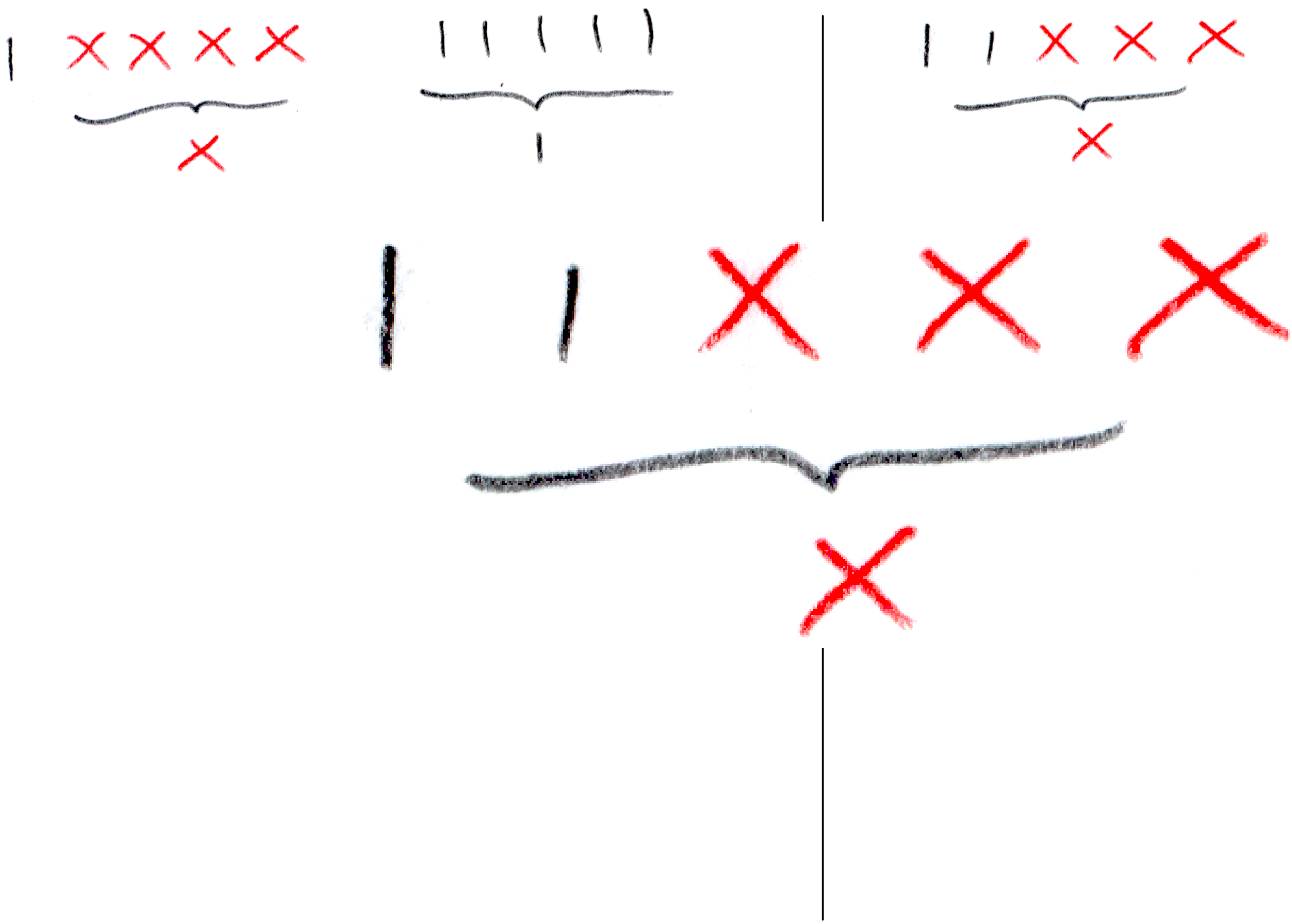
bad

good examples

- **Relative error states** based on ideal recursive decoding
- A **good** block has at most one subblock either in relative error or bad.



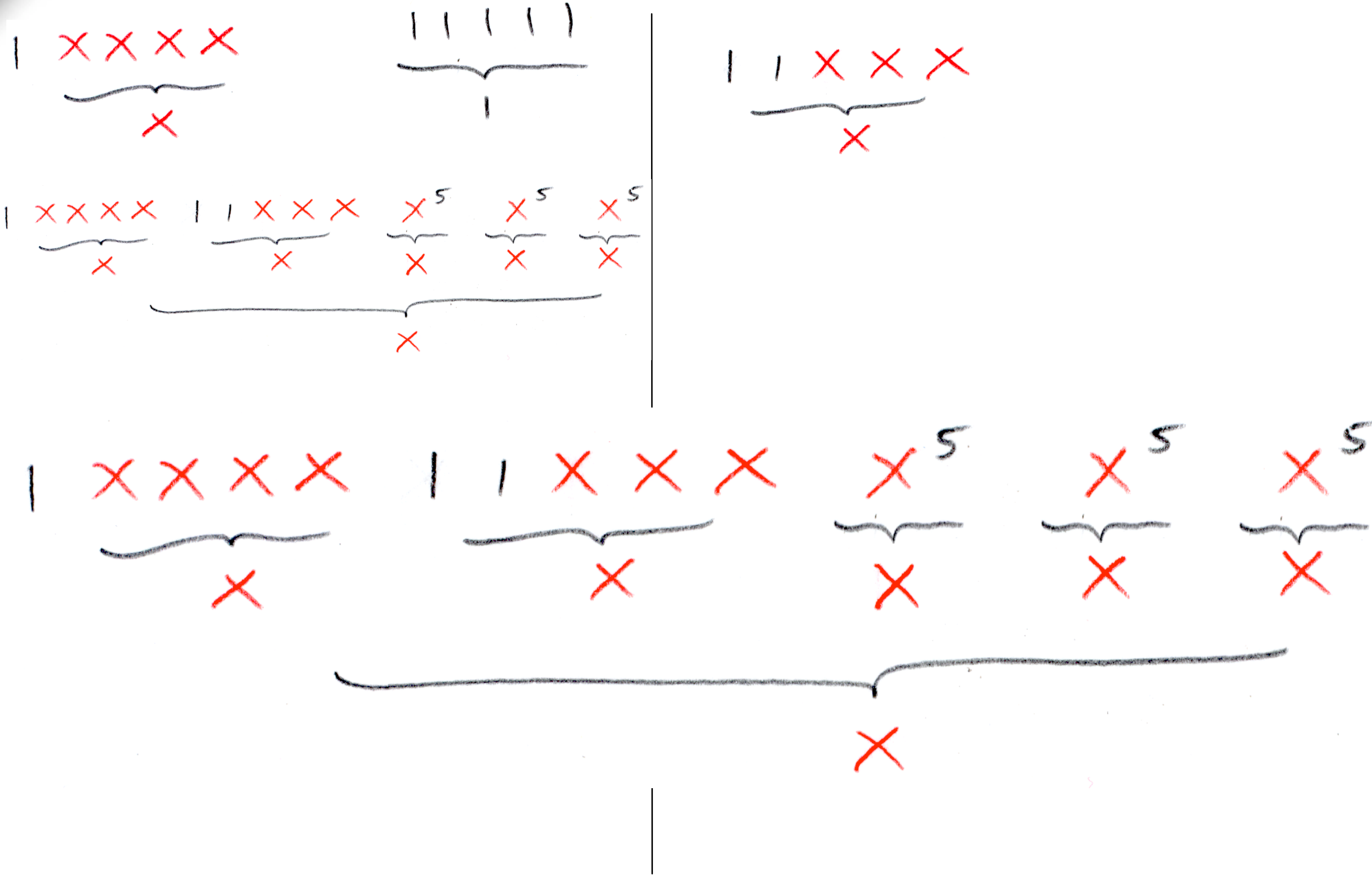
good | bad



good

(at most one subblock either in relative error or bad)

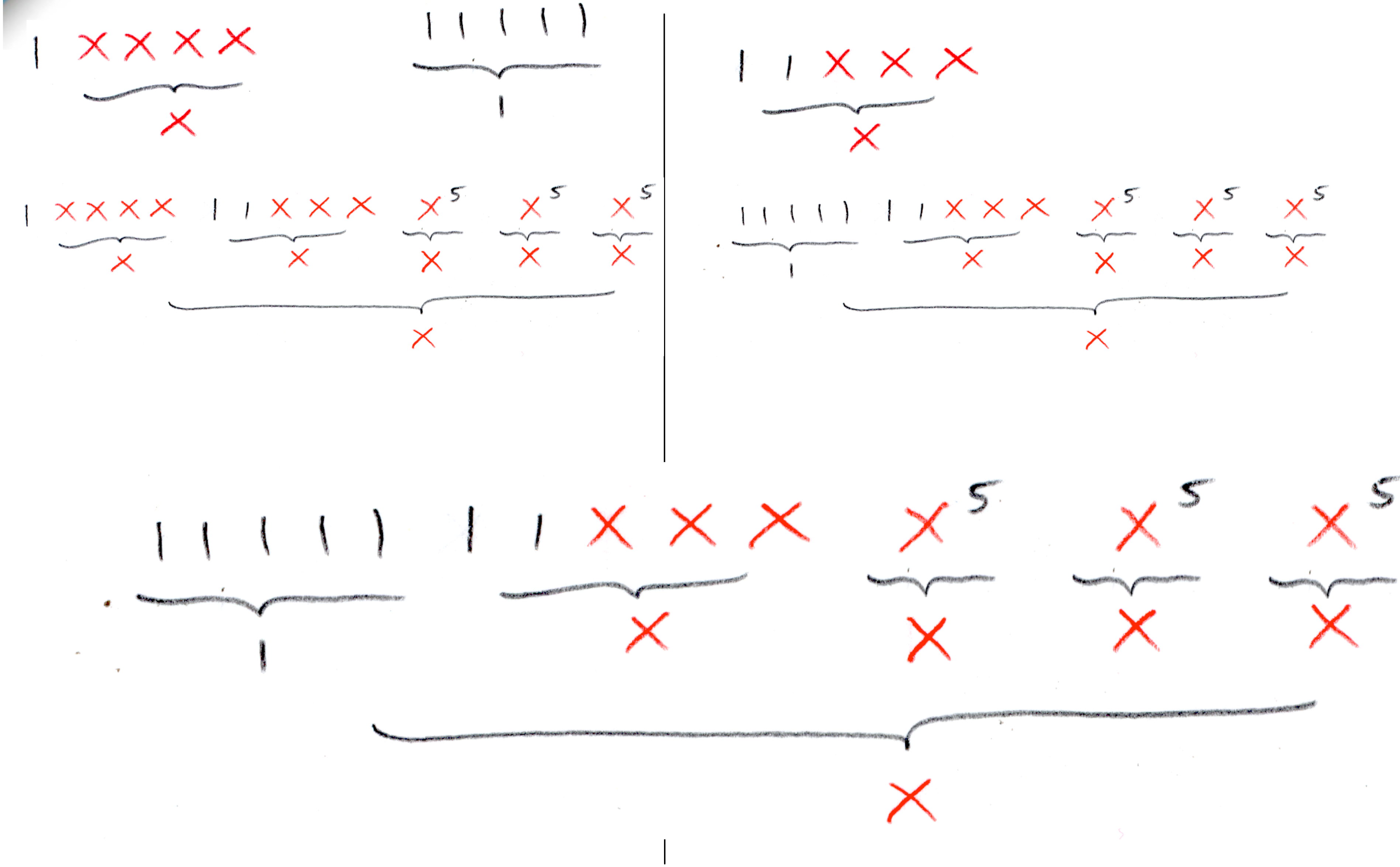
bad



good

(at most one subblock either in relative error or bad)

bad



good

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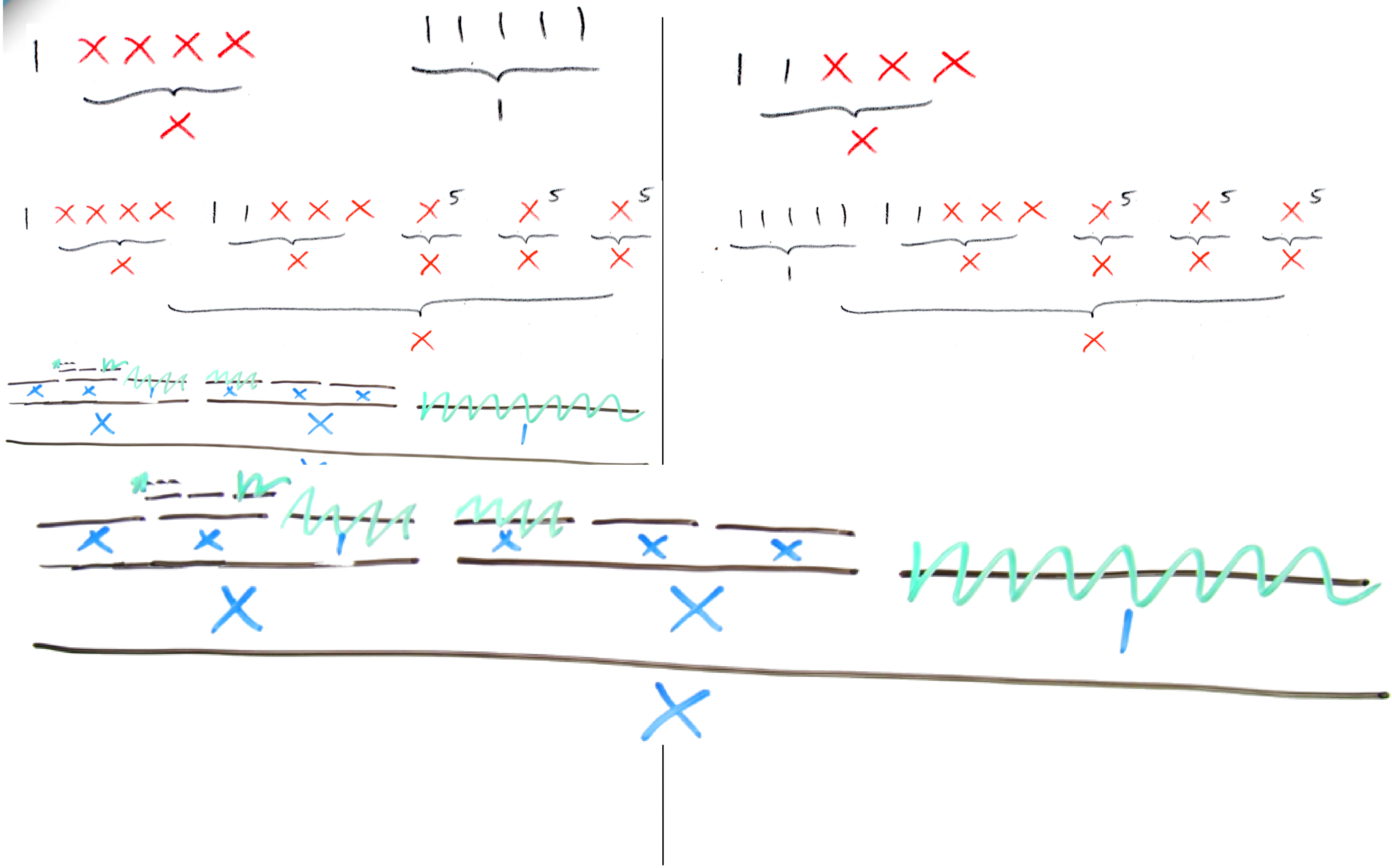
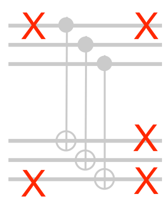
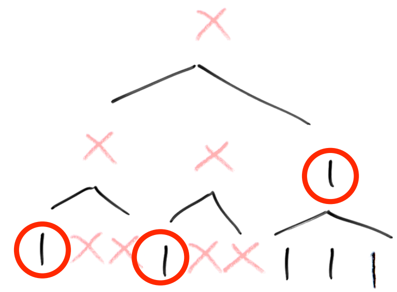


Def: well

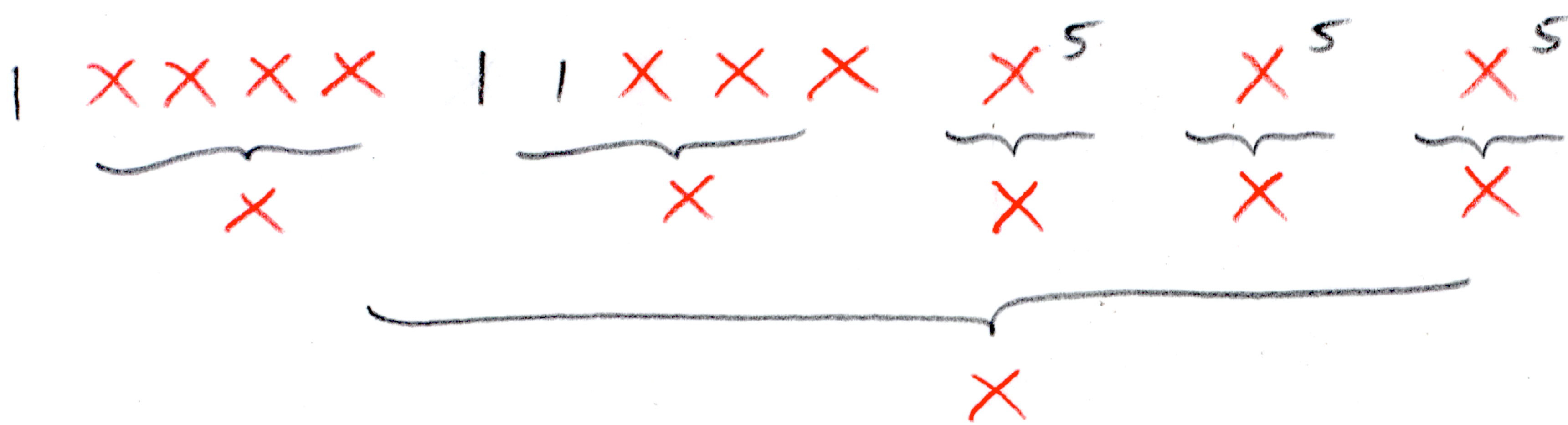
- **Tracking errors**



- **Block error states:** ideal recursive decoding
- **Relative error states**

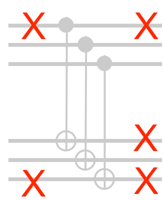


- **Def:** A block_k is good_k if it has at most one subblock_{k-1} either in relative error or not good_{k-1} itself.
(Every bit [$\equiv \text{block}_0$] is good_0.)



Def: well

- **Tracking errors**



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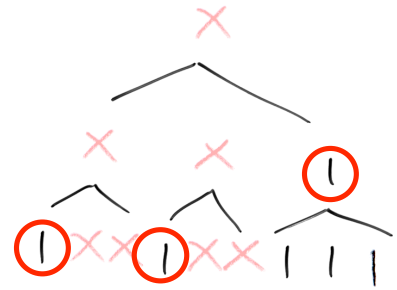
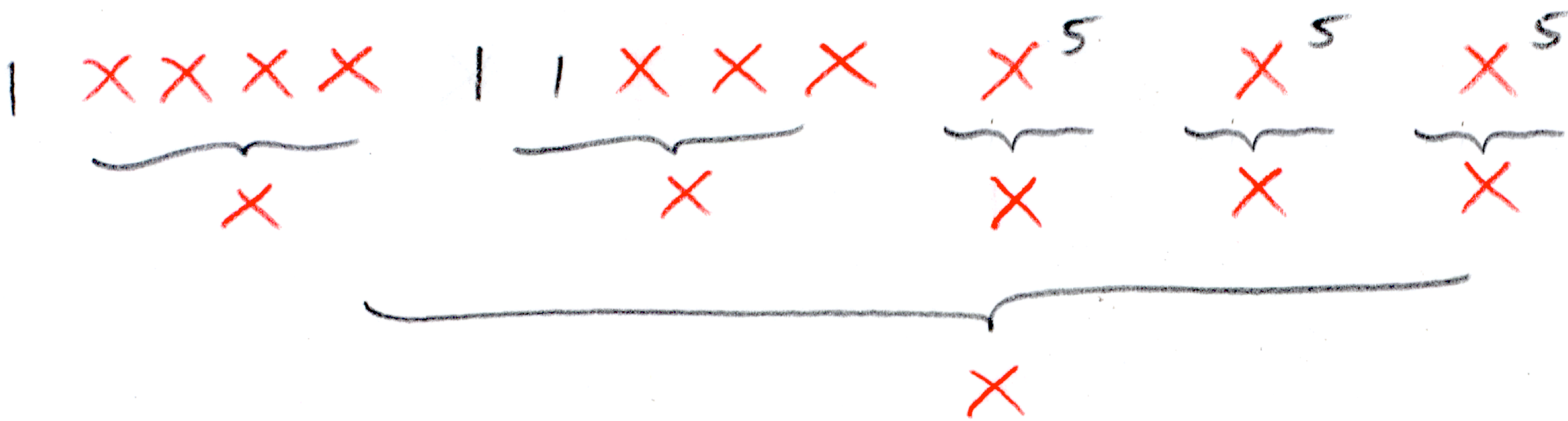
- **Relative error states**



- **Def:** A block_k is $\text{well}_k(p_1,\ldots,p_k)$ if it has at most one subblock_{k-1} either in relative error or not $\text{well}_{k-1}(p_1,\ldots,p_{k-1})$ itself.

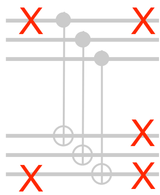
  Additionally, the probability of such a subblock, conditioned on the block's state and the state of all bits in other blocks, is $\leq p_k$.

(Every bit [$\equiv \text{block}_0$] is well_0.)



Def: well

- **Tracking errors**



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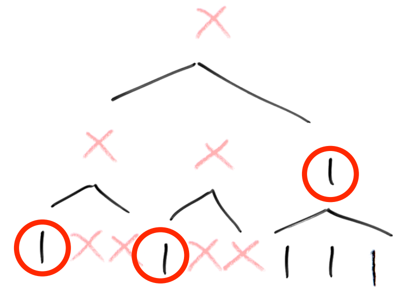
- **Relative error states**



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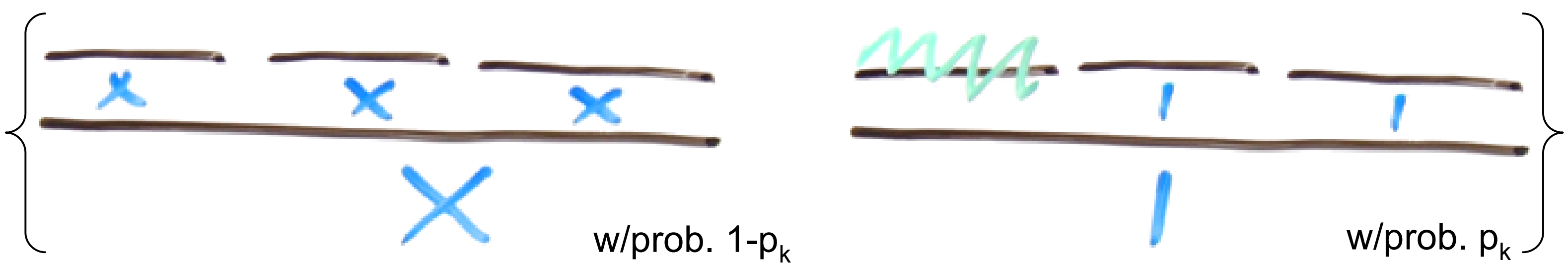
  Additionally, the probability of such a subblock, conditioned on the block's state and the state of all bits in other blocks, is $\leq p_k$.
  (Every bit [$\equiv \text{block}_0$] is well_0.)

- **Note:** *Conditioned* on block's state, e.g.,



is *not* 1-well.

Dist-3 code setup

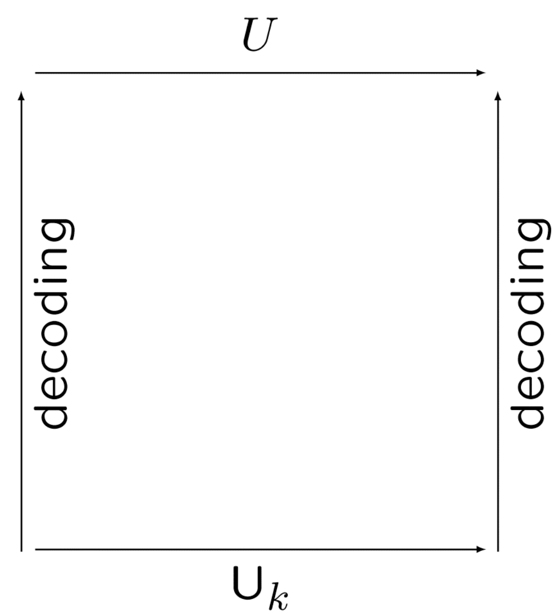
- **Base noise model:** $CNOT_0$ gates fail with X errors independently w/ prob. p

- **Claim C_k** ($CNOT_k$)**:** On success:
  - $Well_k(b_1,\ldots,b_k)$ inputs ⇒ $well_k(b_1,\ldots,b_k)$ outputs, and logical CNOT
  - Arbitrary inputs ⇒ $well_k(b_1,\ldots,b_k)$ outputs, and possibly incorrect logical effect

  Failure prob. ≤ C_k (C_0 = p).

Def: Logical failure

- **Def:** Logical operation U_k on one or more blocks$_k$ has the correct logical effect if the diagram commutes:

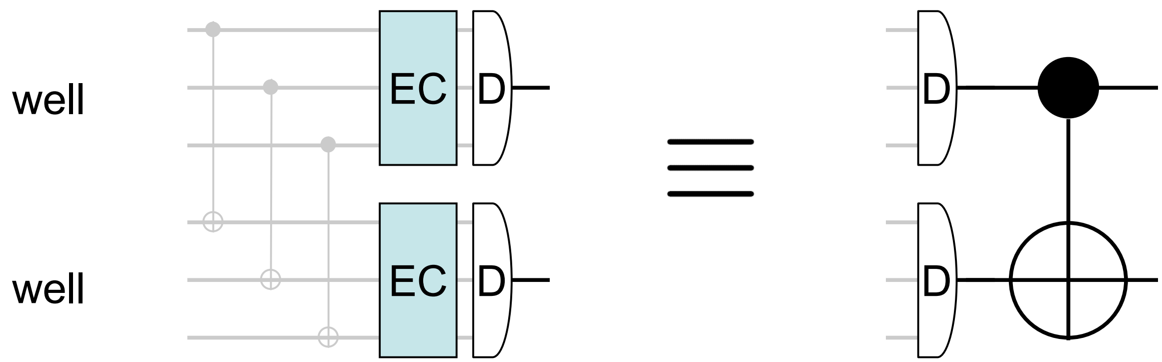


- U_k has a possibly incorrect logical effect if the same diagram commutes but with $P \circ U$ on the top arrow, where P is a Pauli operator or Pauli product on the involved blocks.

Dist-3 code setup

- **Claim C_k** (CNOT$_k$)**:** On success:
  - Well inputs ⇒ well outputs, and logical CNOT
  - Arbitrary inputs ⇒ well outputs

Failure prob. ≤ C_k (C_0 = p).



- **Claim B_k** (Correction$_k$)**:** On success:
  - Well$_k$(b_1,…,b_k) input ⇒ well$_k$(b_1,…,b_k) output, and no logical effect
  - Arbitrary input ⇒ well$_k$(b_1,…,b_k) output

Failure prob. ≤ B_k (B_0 = 0).

Additionally, if all but one of the input subblocks$_{k-1}$ are well$_{k-1}$(b_1,…,b_{k-1}), then with probability at least 1-B_k' there is no logical effect and the output is well$_k$(b_1,…,b_k).

Dist-3 code threshold proof

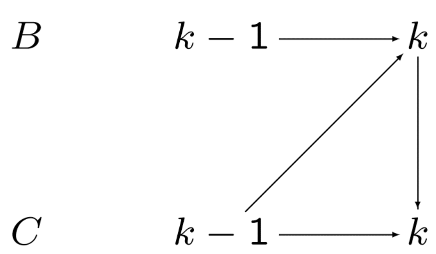
- **Two operations:**

  B. Error correction
  C. (Logical) CNOT gate

- **Two indexed claims:**

  B_k Error correction$_k$ success except w/ prob. $\leq B_k$
  C_k CNOT$_k$ success except w/ prob. $\leq C_k$

- **Proofs by induction:** Implications:



$$B_k = O\left((B_{k-1} + C_{k-1})^2\right)$$

$$C_k = O\left(B_k + C_{k-1}^2\right)$$

Dist-3 code threshold proof

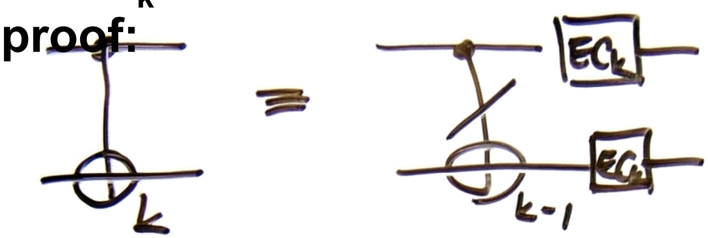
- **Claim B_k** (Correction$_k$)**:** On success:
  - Well$_k$(b_1,…,b_k) input ⇒ well$_k$(b_1,…,b_k) output, no logical effect
  - Arbitrary input ⇒ well$_k$(b_1,…,b_k) output

  Failure prob. ≤ B_k (B_0 = 0).

  Additionally, if all but one of input subblocks$_{k-1}$ are well$_{k-1}$(b_1,…,b_{k-1}), then w/ prob. ≥ 1-B_k', output is well$_k$(b_1,…,b_k) and no logical effect.
- **Claim C_k** (CNOT$_k$)**:** On success:
  - Well inputs ⇒ well outputs, and logical CNOT
  - Arbitrary inputs ⇒ well outputs

  Failure prob. ≤ C_k (C_0 = p).

CNOT$_k$ proof:



- Assume input blocks are well$_k$(b_1,…,b_k). Declare failure if either Correction$_k$ fails, or if there are two level k-1 failures.

$$C_k \equiv \Big(2B_k + (nC_{k-1})(2B'_k) + \binom{n}{2} C_{k-1}^2\Big) + 2b_k(2B'_k + nC_{k-1}) + b_k^2$$

- On success, transverse CNOTs$_{k-1}$ implement the correct logical effect (but possibly correlate errors). The successful Corrections$_k$ have no logical effect but restore wellness (bounded dependencies).

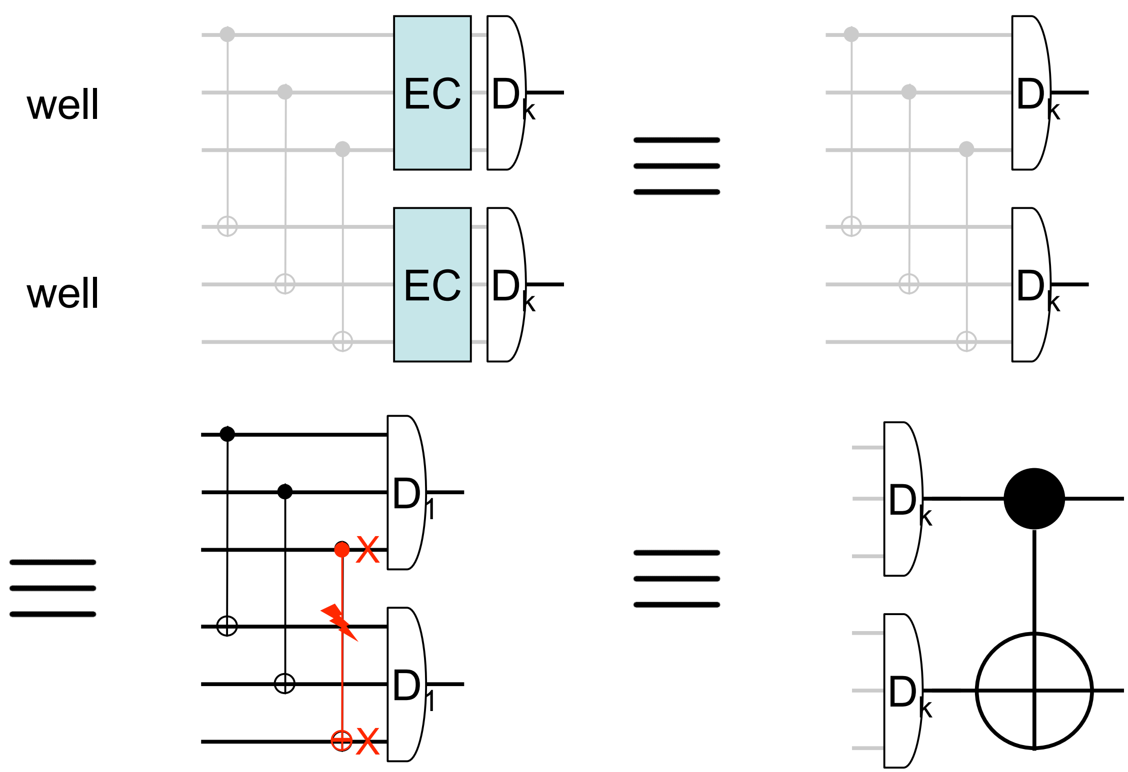
Dist-3 code threshold proof

- **Claim C_k** (CNOT$_k$)**:** On success:
  - Well inputs ⇒ well outputs, and logical CNOT
  - Arbitrary inputs ⇒ well outputs

  Failure prob. ≤ C_k (C_0 = p).

CNOT$_k$ proof: Failure if either Correction$_k$ fails, or if there are two level k-1 failures.

Success: transverse CNOTs$_{k-1}$ implement correct logical effect. Corrections$_k$ have no logical effect.

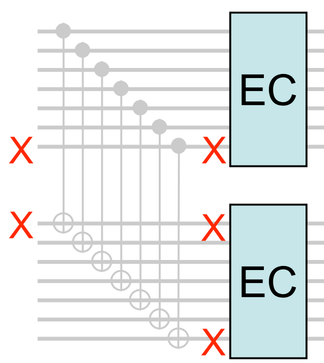


□

Aliferis-Gottesman-Preskill threshold intuition

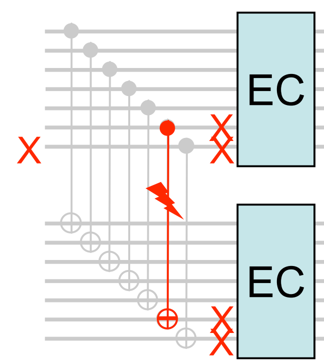
- **Aharonov & Ben-Or Idea:** Maintain inductive invariant of (1-)goodness. (A block is good “if it has at most one bad subblock.”)
- Two ways it can fail with distance-three codes:

1.



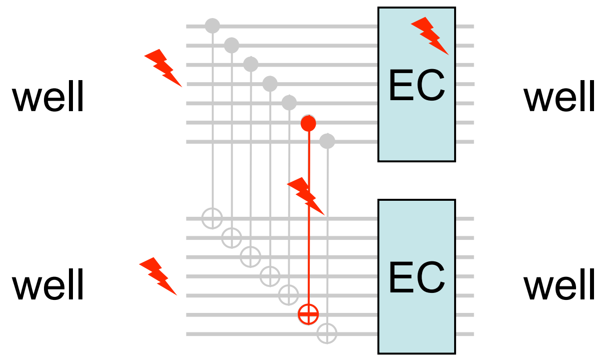
Both input blocks have a bad subblock.

2.



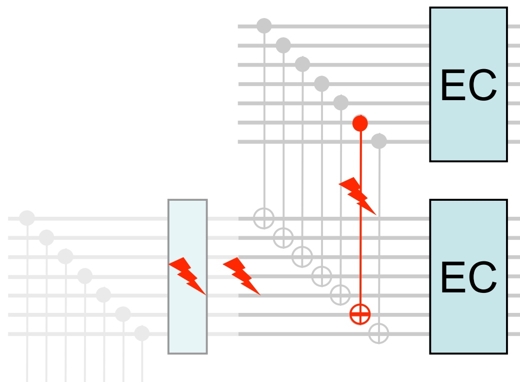
One input block has a bad subblock, and an additional error occurs.

Aliferis-Gottesman-Preskill threshold intuition



- A/B: Maintain ‘good’ness — two faults in rectangle cause logical failure (d≥5)
- R: Maintain ‘well’ness — two faults in rectangle or well input cause logical failure

Aliferis-Gottesman-Preskill threshold intuition



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…errors in input come from errors in
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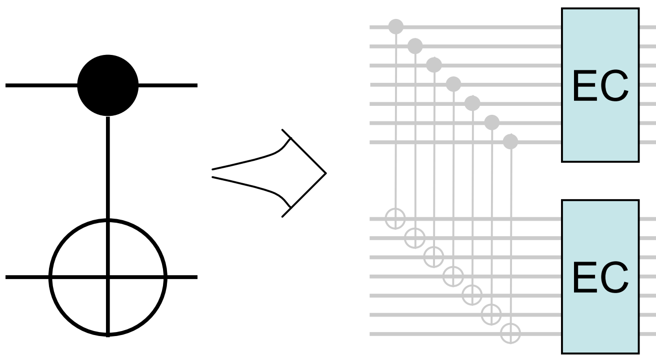
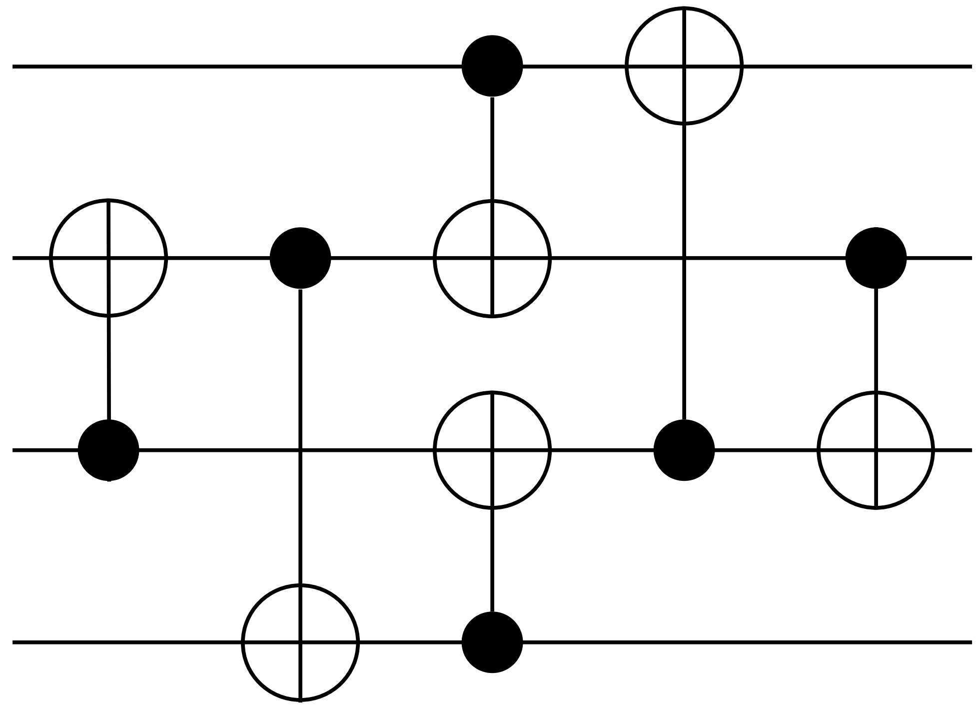
- A/G/P: two faults in *extended* (overlapping) rectangle cause logical failure

Aliferis-Gottesman-Preskill threshold intuition





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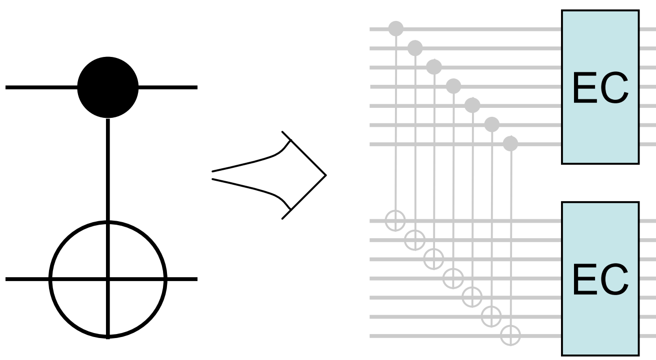

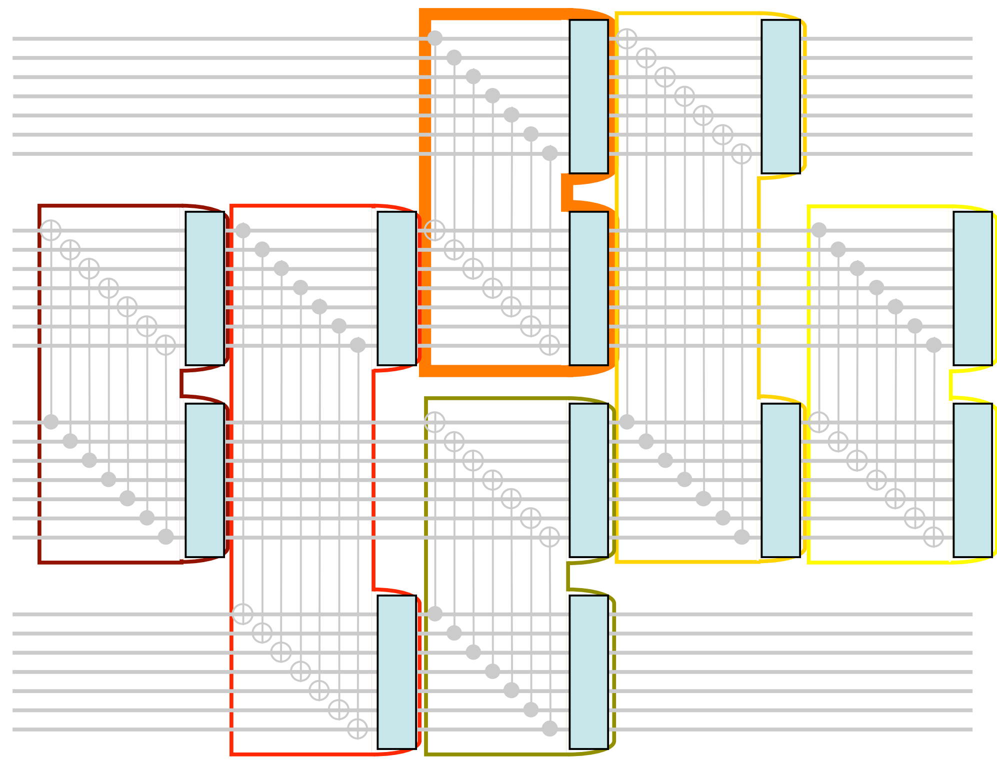

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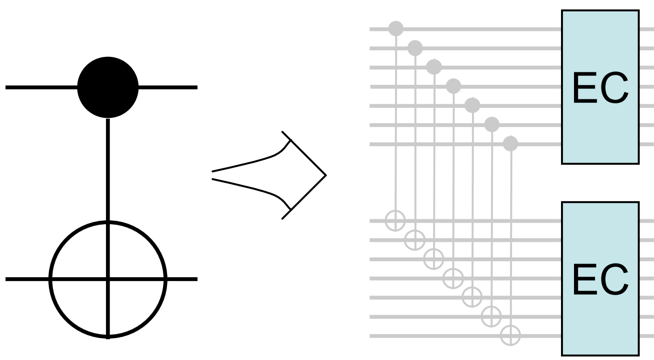
Aliferis-Gottesman-Preskill threshold intuition



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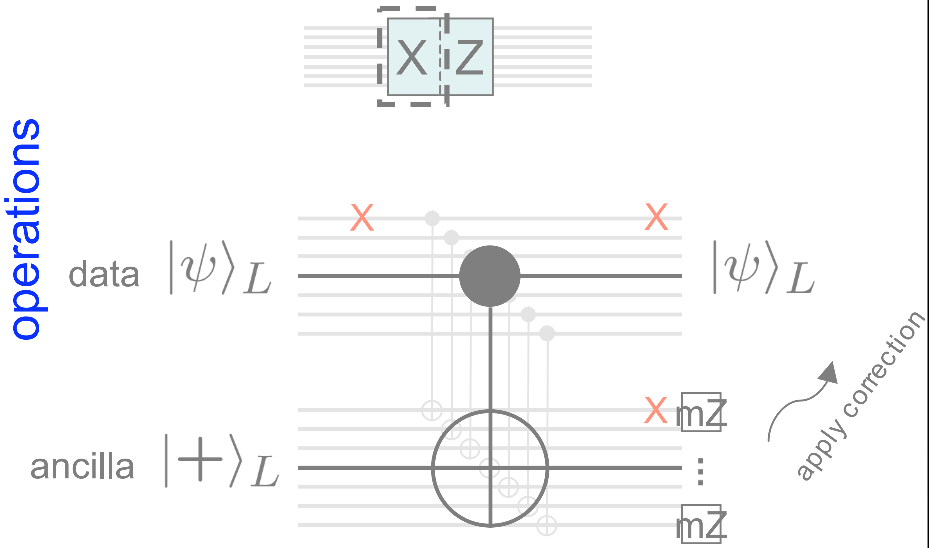
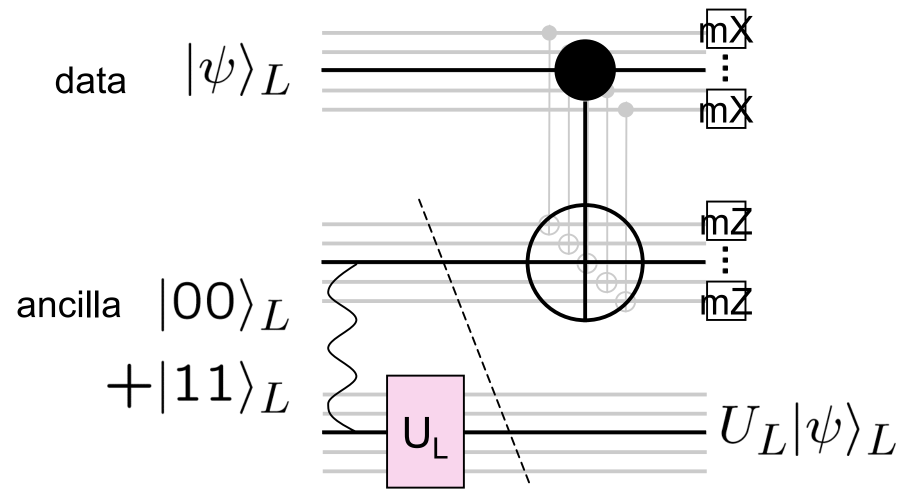
Steane-type error correction

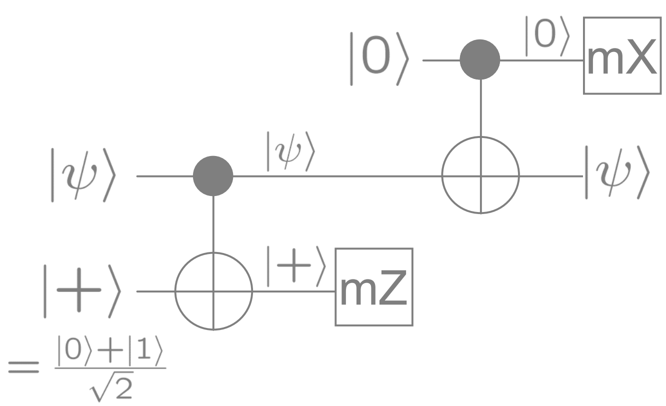
Knill-type correction + computation

Physical operations

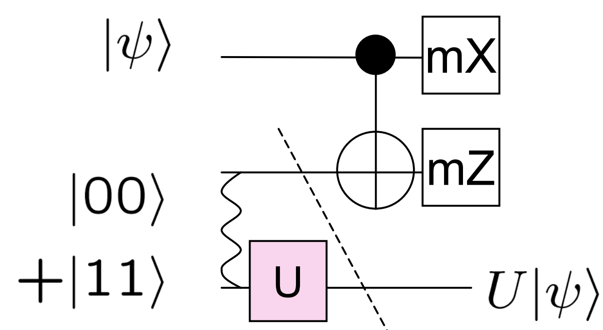




Logical operations

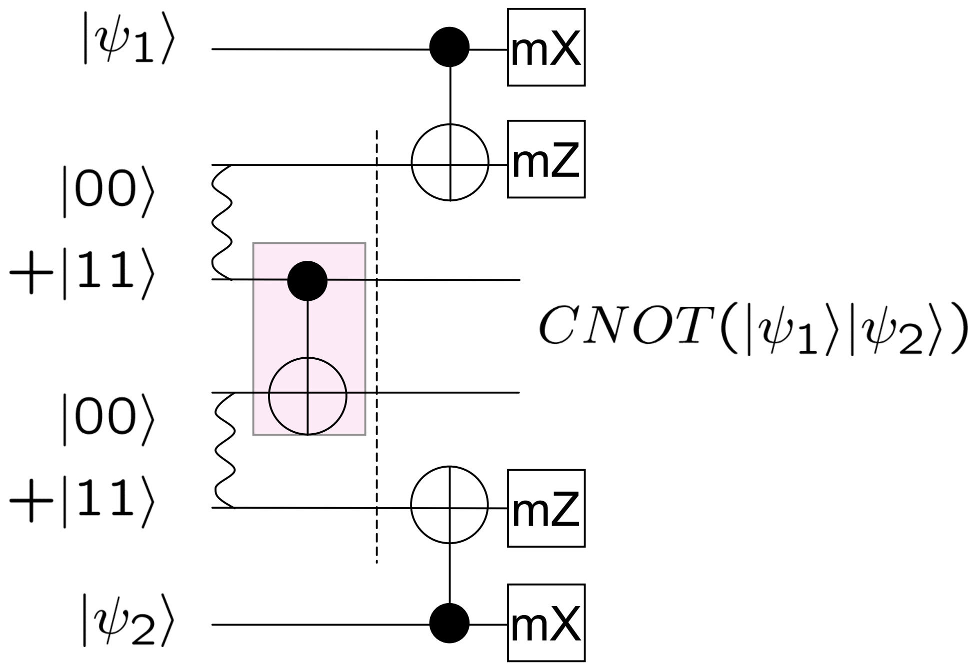


Teleportation



Teleporting a CNOT gate

Logical operations



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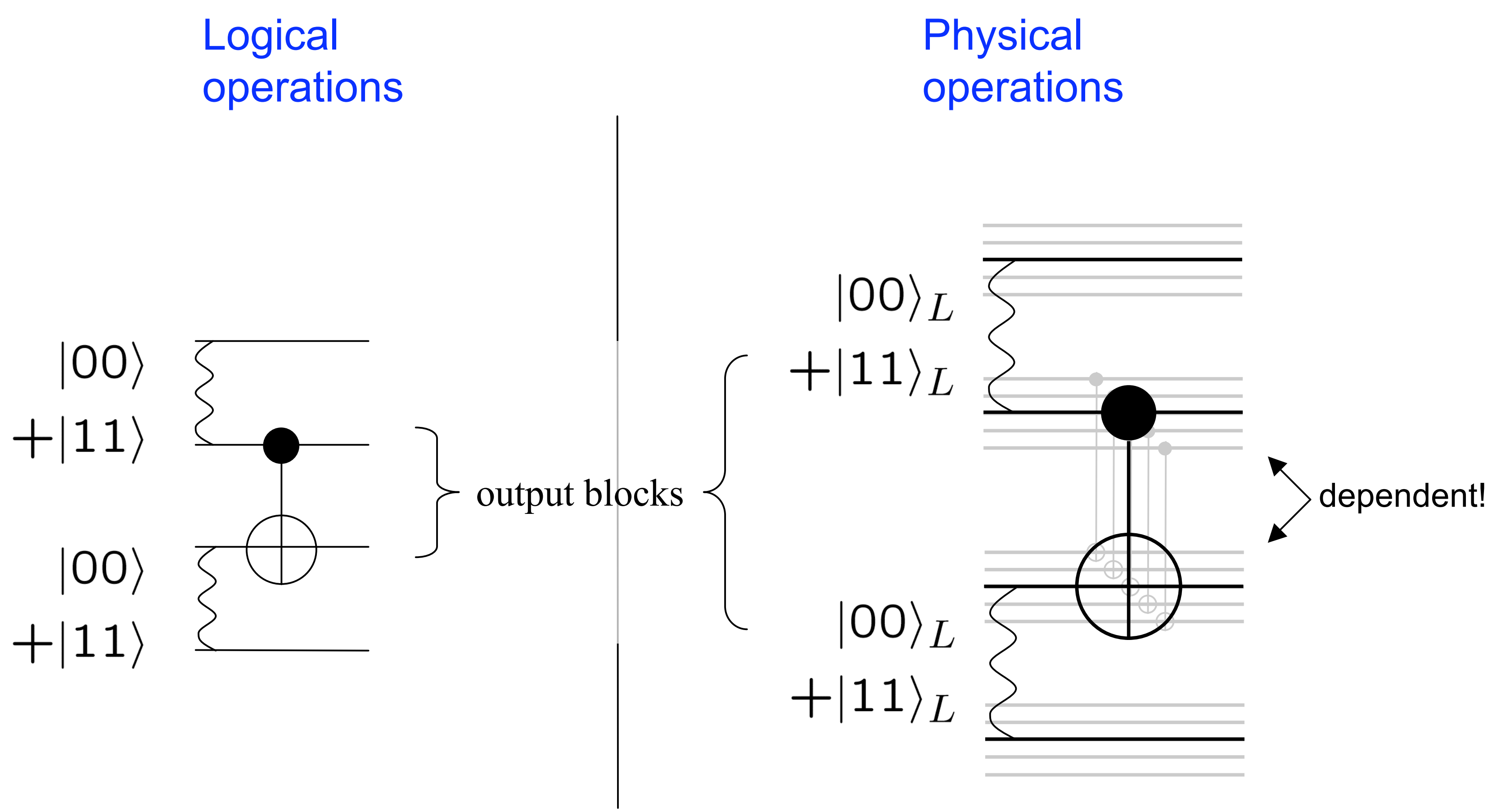

Teleporting a CNOT gate

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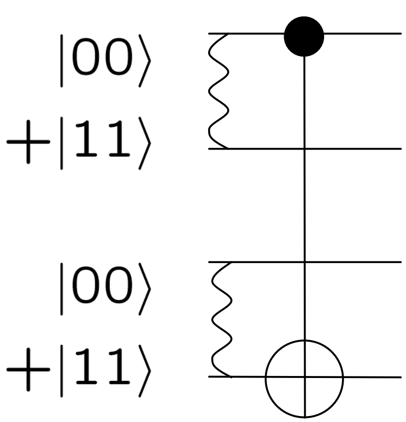
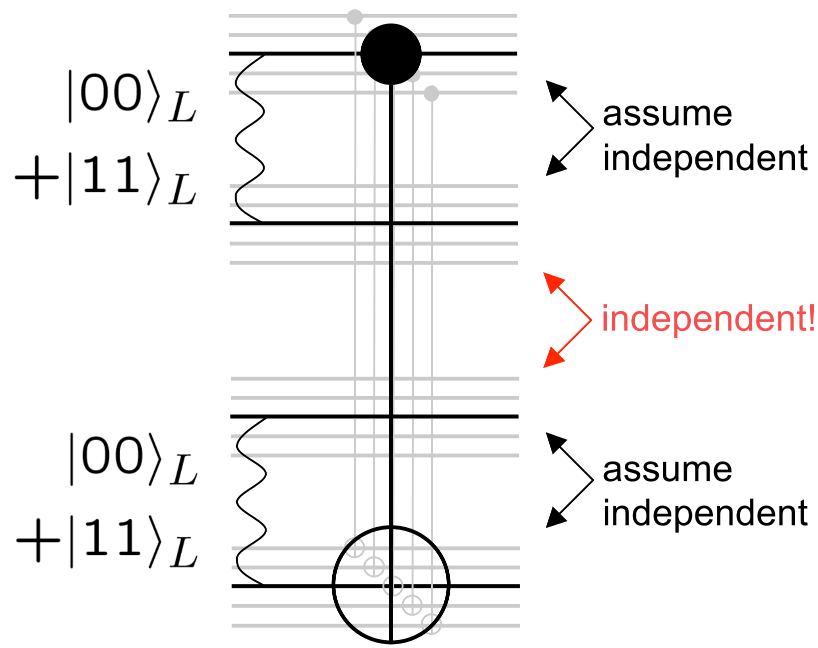


Physical operations



$\Rightarrow$ Achieving independent errors on CNOT output blocks reduces to preparing encoded Bell states with block-independent errors

Unfortunately, this is impossible… But:

Summary

- New threshold proof
  - Based on bounding the *distribution* of errors in the system at each time step
  - More efficient than classical threshold proofs, leads to higher rigorous noise threshold lower bounds
  - Works for concatenated distance-three codes
- Possible extensions
  - Improved analysis of optimized standard fault-tolerance schemes

    (Ouyang, R.: 10^{-4})
  - Extend proof to work with schemes using distance-two codes and extensive postselection. Major difficulty is obtaining better control over error distribution, particularly of dependencies and of errors in the *bad* blocks.

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