A fault-tolerant one-way quantum computer

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Main idea

Universal quantum computation by local measurements:

- A three-dimensional cluster state is a *fault-tolerant fabric*.

2D cluster state → **3D cluster state**

Retain universality,
add fault-tolerance

Make use of geometry!

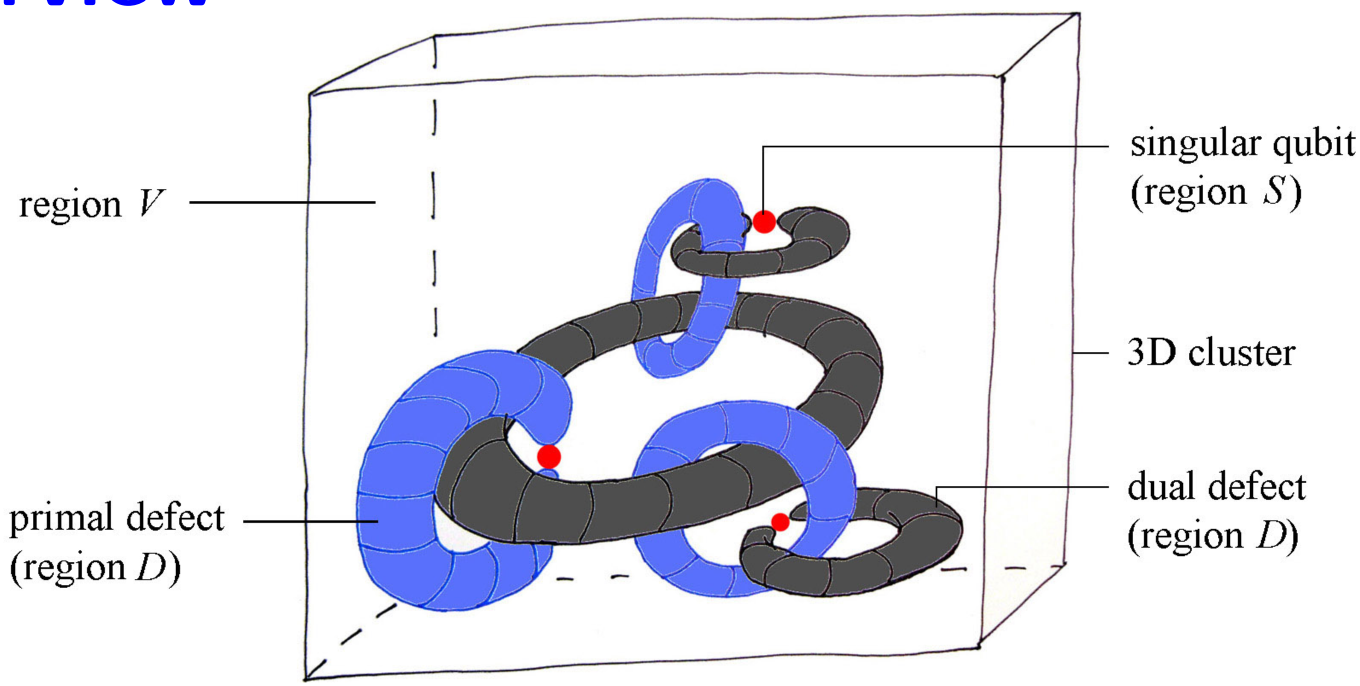
Result

- Geometric constraint: only *local* and *next-neighbor* interaction of qubits on a three-dimensional lattice required.

- Fault-tolerance threshold of 1.1×10^{-3} (each source).
  Error sources: preparation, gates, storage and measurement.

- Polynomial overhead.

Overview



Three cluster regions:

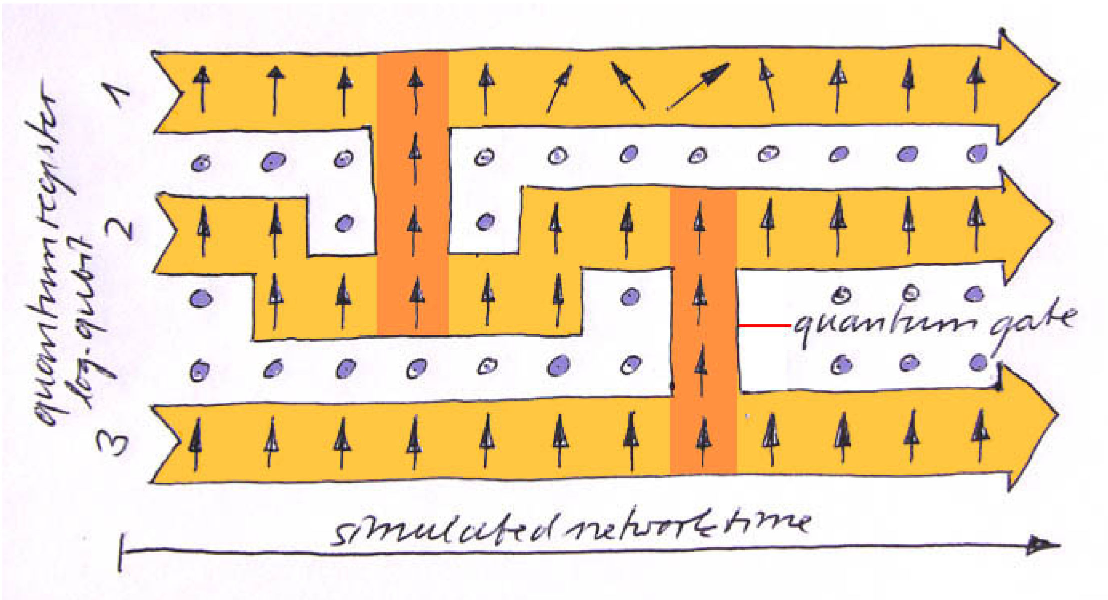
V (Vacuum), D (Defect) and S (Singular qubits).

Qubits $q \in V$: local X-measurements,
Qubits $q \in D$: local Z-measurements,
Qubits $q \in S$: local measurements of $\frac{X \pm Y}{\sqrt{2}}$.

Preliminiaries:

The one-way quantum computer, cluster states and topological codes

The one-way quantum computer



measurement of Z ($\odot$), X ($\uparrow$), $\cos\alpha\, X + \sin\alpha\, Y$ ($\nearrow$)

- Universal computational resource: cluster state.
- Information written onto the cluster, processed and read out by one-qubit measurements only.

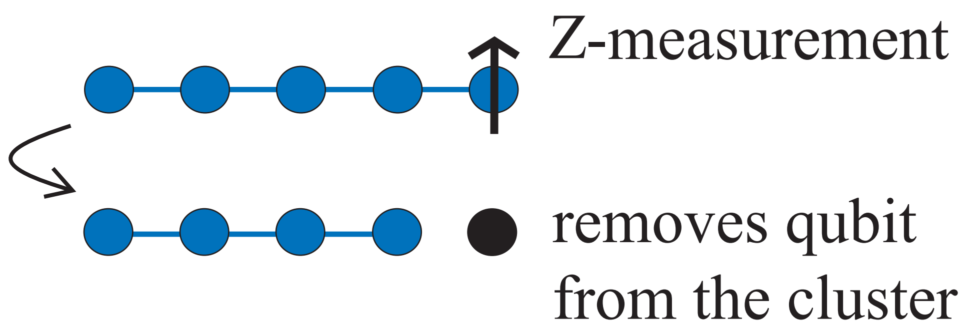
Cluster states

A cluster state $|\phi\rangle_{\mathcal{C}}$ on a cluster $\mathcal{C}$ is the single common eigenstate of the stabilizer operators

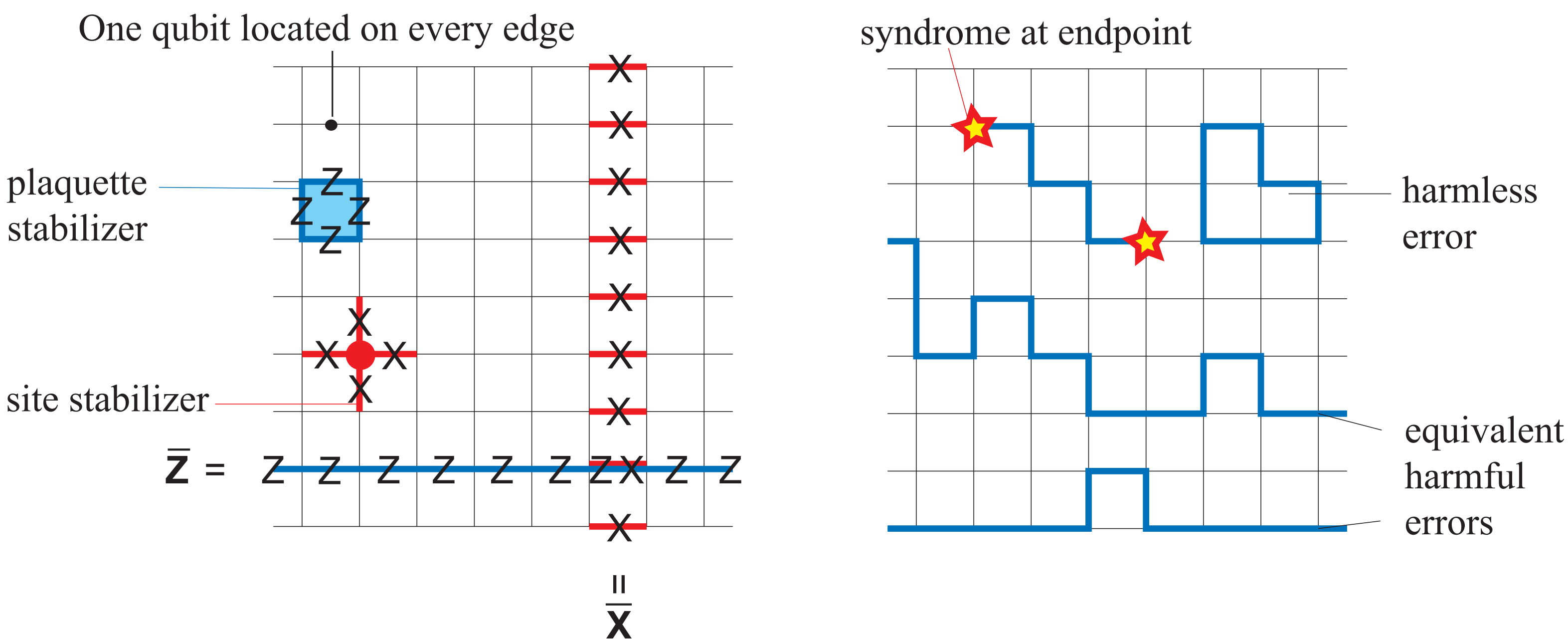
$$K_a = X_a \bigotimes_{b \in N(a)} Z_b, \quad \forall a \in \mathcal{C}, \tag{1}$$

where $b \in N(a)$ if a,b are spatial next neighbors in $\mathcal{C}$.

Z-**Rule:**



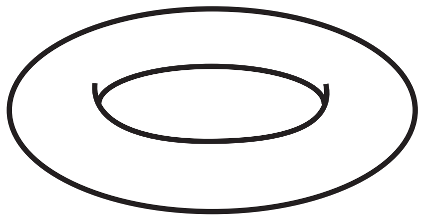
Topological quantum codes



- Errors are represented by chains.
- Homologically equivalent chains correspond to physically equivalent errors.
- Harmfull errors stretch across the entire lattice (rare events).

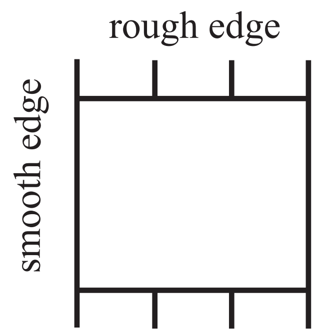
A. Kitaev,quant-ph/9707021 (1997).

Topological quantum codes



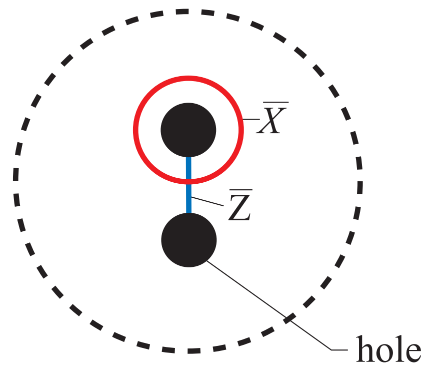
Torus

2 Qubits



Plane segment

1 Qubit

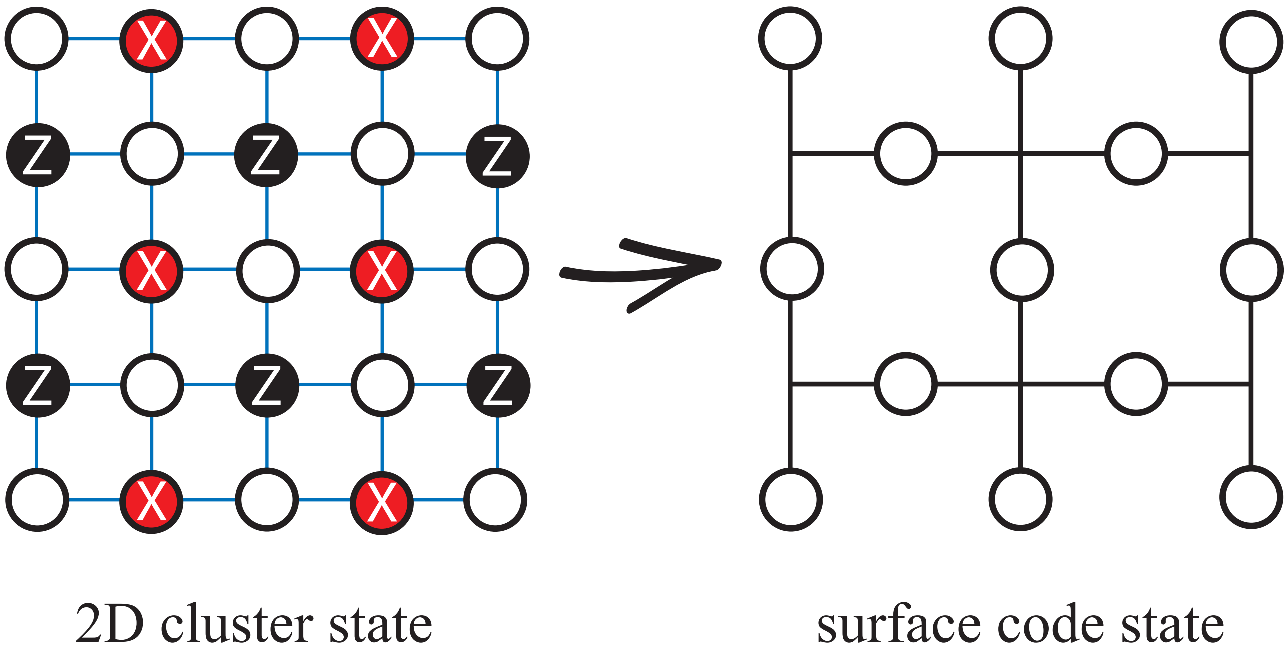


Plane with 2 holes

1 Qubit

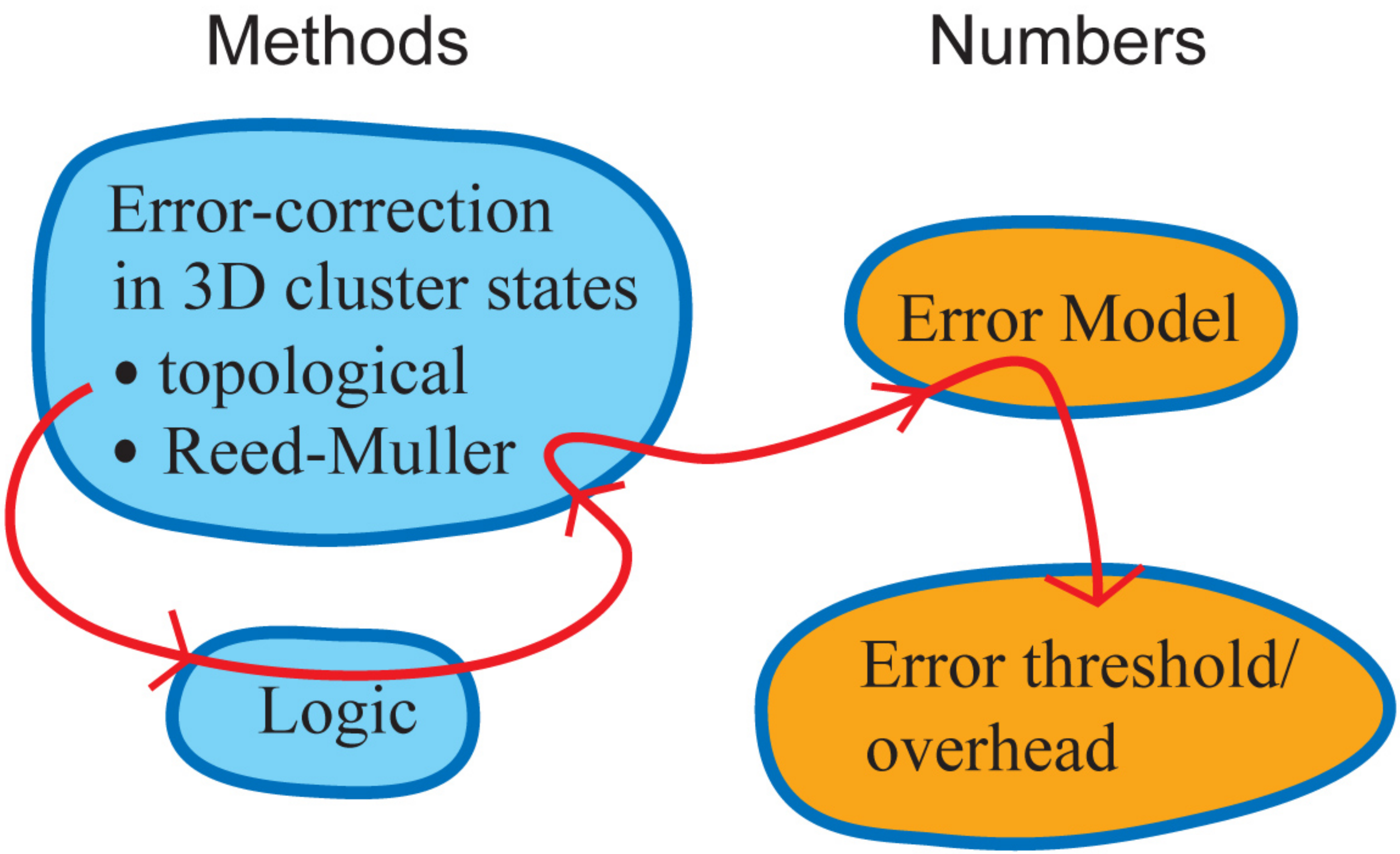
- Storage capacity of the code depends upon the topology of the code surface.

Link



- Obtain surface code state from 2D cluster state via regular pattern of Z- and X-measurements.

Talk outline

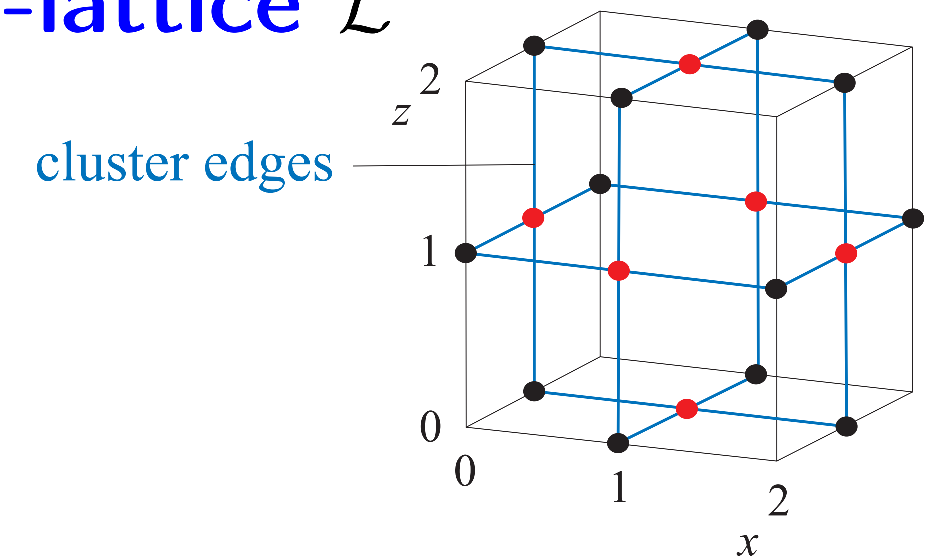


Part I:

Error correction in 3D cluster states

Cluster states in three spatial dimensions provide intrinsic topological error correction.

Cluster $\mathcal{C}$ and bcc-lattice $\mathcal{L}$



elementary cell of $\mathcal{L}$

qubit location: (even, odd, odd) - face of $\mathcal{L}$,

qubit location: (odd, odd, even) - edge of $\mathcal{L}$,

syndrome location: (odd, odd, odd) - cube of $\mathcal{L}$,

syndrome location: (even, even, even) - site of $\mathcal{L}$.

Topological error correction in V

Measurement pattern:

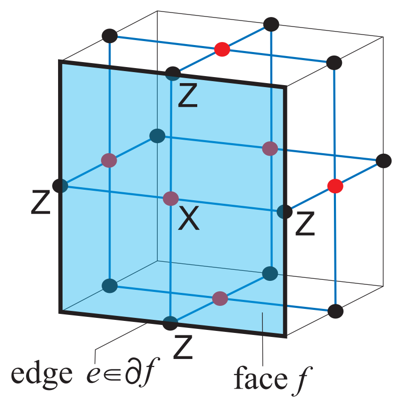
- The qubits $q \in V$ are individually measured in the X-basis.

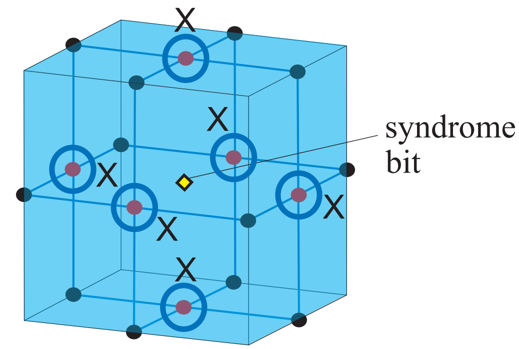
Errors:

- Consider probabilistic Pauli errors.
- Sufficient to consider Z-errors.

  (X-errors are absorbed into the X-measurement, $\frac{I \pm X}{2} X = \pm \frac{I \pm X}{2}$.)

Homology





- Stabilizer elements associated with faces f of $\mathcal{L}$:

$$K(f) = \bigotimes_{a \in f} X_f \bigotimes_{b \in \partial f} Z_b. \tag{2}$$

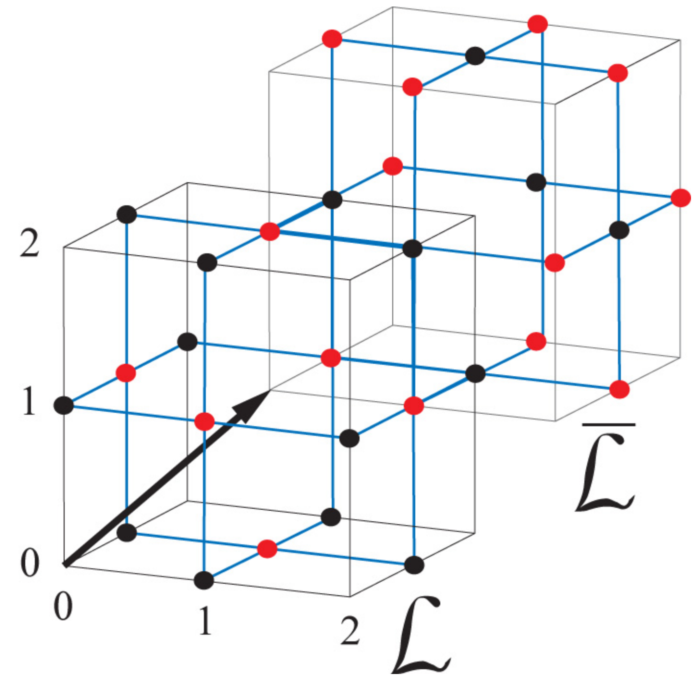
- Stabilizer for syndrome ($[K(f), X_q] = 0 \; \forall \, q \in V$):

$$\partial f = 0. \tag{3}$$

- One syndrome bit per cell of $\mathcal{L}$. Protects the face qubits.

What about the edge qubits?

Lattice duality $\mathcal{L} \longleftrightarrow \overline{\mathcal{L}}$



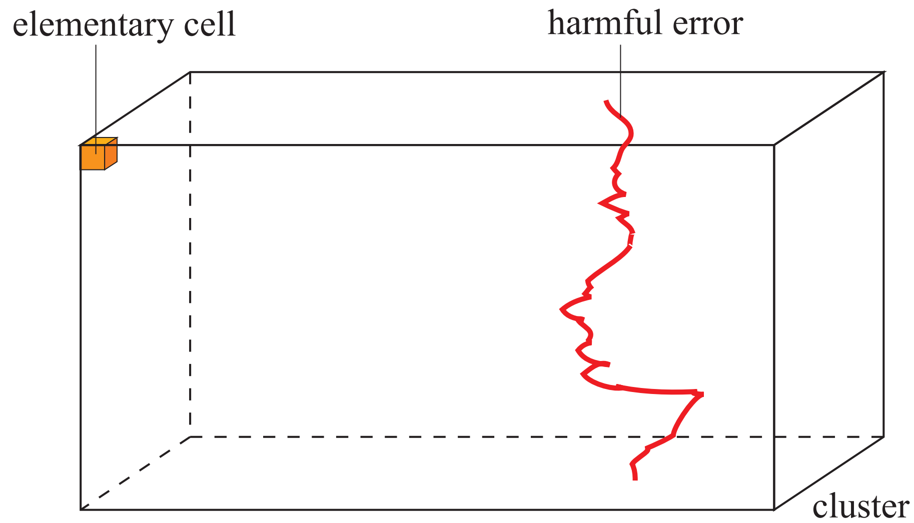
Translation by vector $(1,1,1)^T$:

- Cluster $\mathcal{C}$ invariant,
- $\mathcal{L}$ (primal) $\longrightarrow \overline{\mathcal{L}}$ (dual).

$$\begin{aligned} \text{face of } \mathcal{L} &\; - \; \text{edge of } \overline{\mathcal{L}}, \\ \text{edge of } \mathcal{L} &\; - \; \text{face of } \overline{\mathcal{L}}, \\ \text{site of } \mathcal{L} &\; - \; \text{cube of } \overline{\mathcal{L}}, \\ \text{cube of } \mathcal{L} &\; - \; \text{site of } \overline{\mathcal{L}}, \end{aligned} \tag{4}$$

- Edge qubits protected by stabilizer on dual lattice $\overline{\mathcal{L}}$.
- Many objects in this scheme exist as ‘primal’ and ‘dual’.

Topological error correction in V



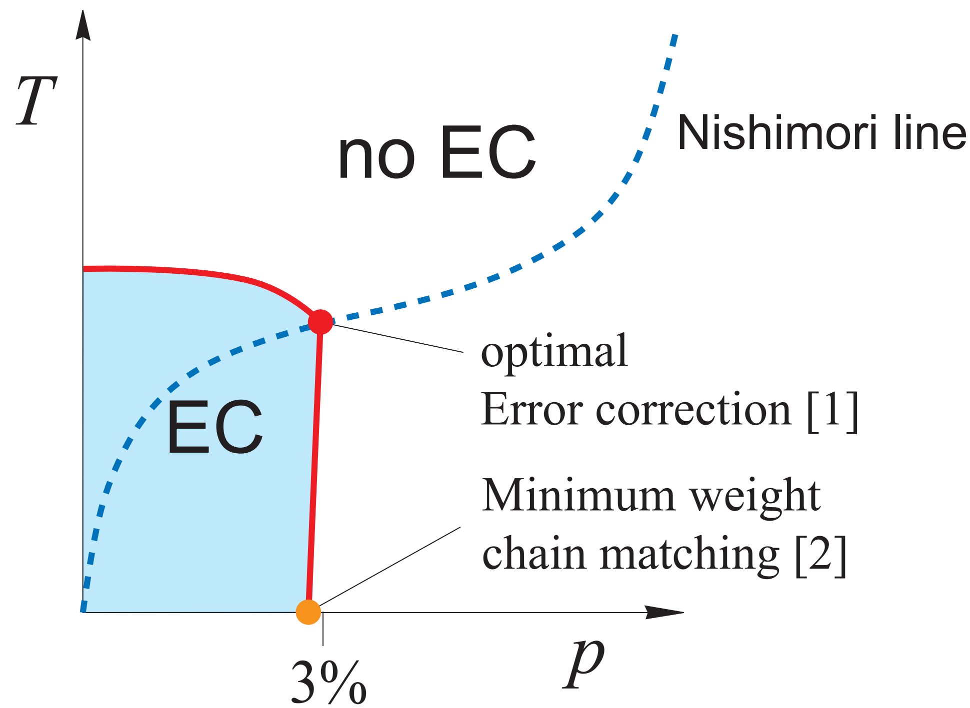
- One syndrome bit for each elemetary cell of $\mathcal{L}$.
- Harmful errors stretch across entire lattice $\mathcal{L}$.

-> Leads to *Random plaquette Z_2-gauge model* (RPGM) [1].

[1] Dennis et *al.*, quant-ph/0110143 (2001).

RPGM: schematic phase diagram

Map error correction to statistical mechanics:



[1] T. Ohno et *al.*, quant-ph/0401101 (2004).

[2] E. Dennis et *al.*, quant-ph/0110143 (2001); J. Edmonds, Canadian J. Math. **17**, 449 (1965).

Cluster region V

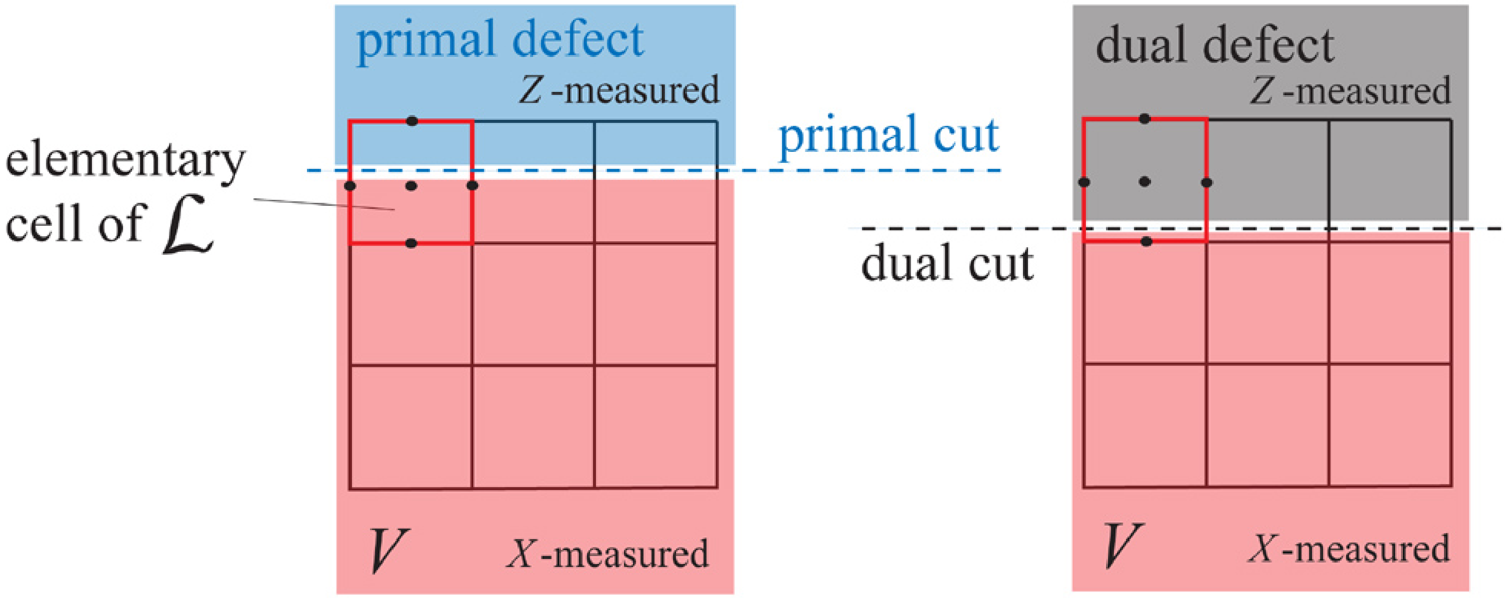
Defects $d \in D$

Singular qubits

Part II:
Quantum Logic

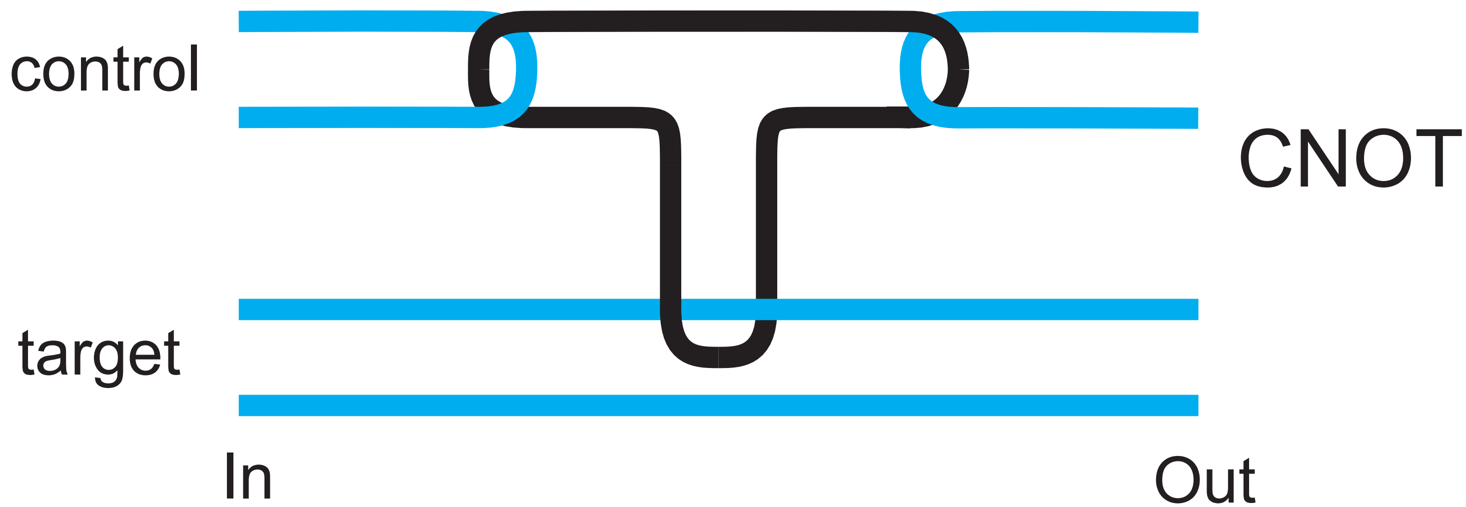
Fault-tolerant quantum logic is realized via topologically entangled engineered lattice defects.

Defects



- Defects are regions of the cluster where qubits are measured in the Z-basis.
- Defects create cluster boundaries (cuts).
- There are *primal* and *dual* defects.

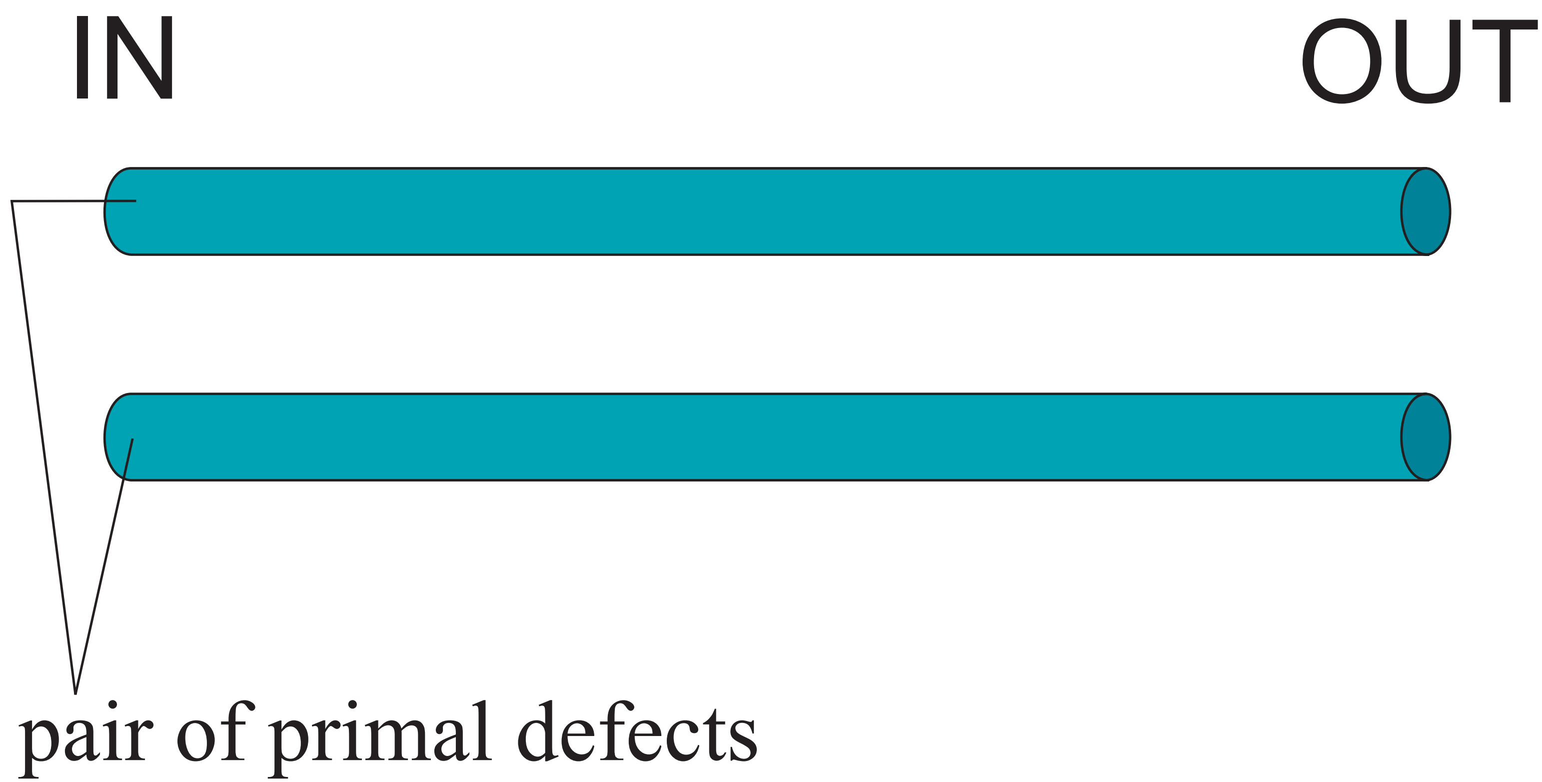
Defects for quantum logic



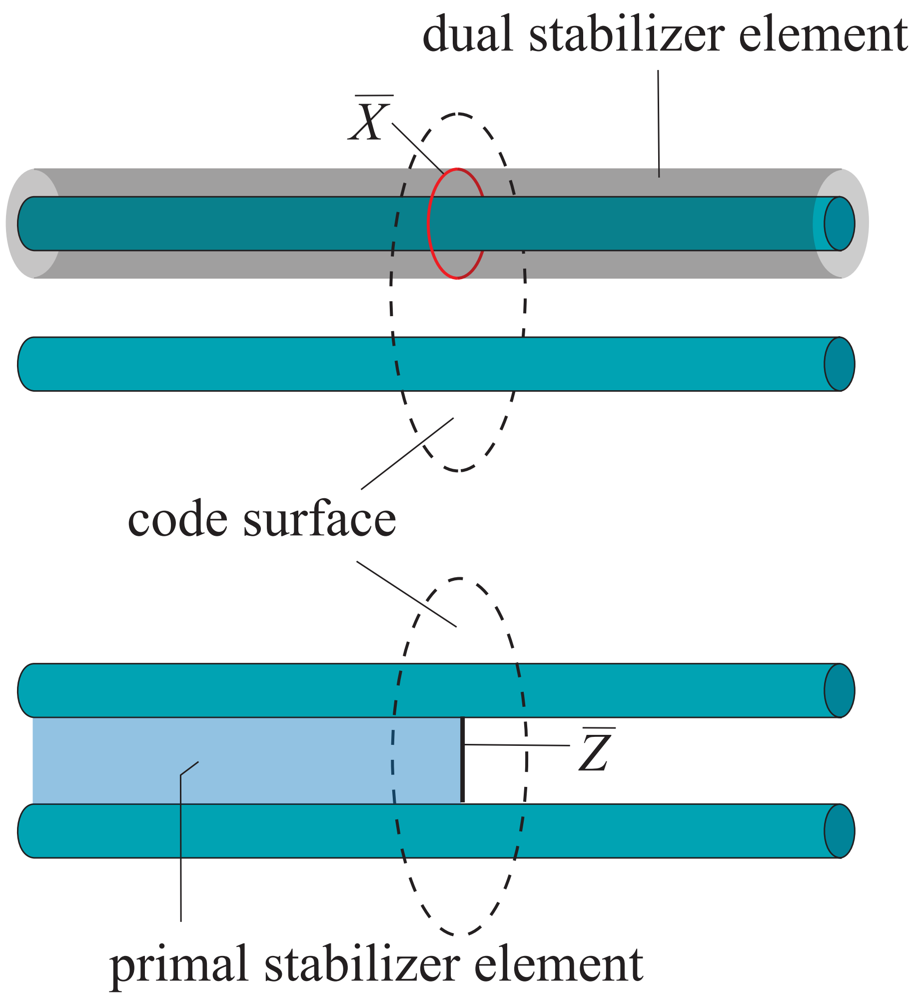
A quantum circuit is encoded in the way primal and dual defects are wound around another.

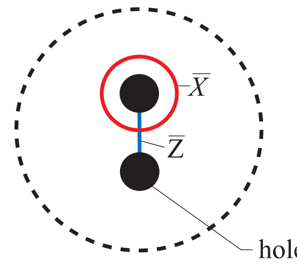
Quantum gates, Part I

Piece of wire



Quantum gates, Part I

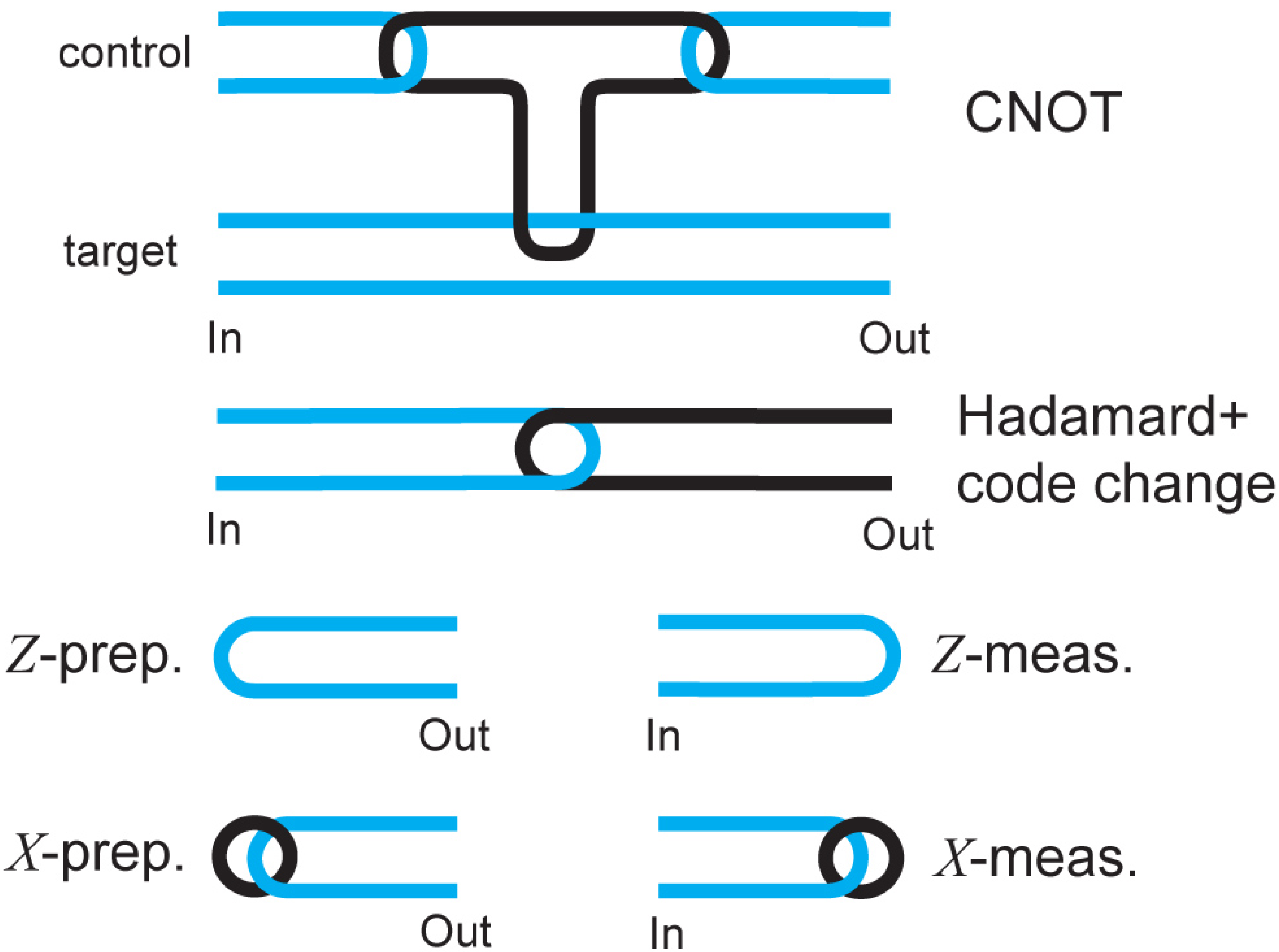




Plane with 2 holes

1 Qubit

Quantum gates, Part I



- Displayed fault-tolerant gates are not universal.
- Need one non-Clifford element:
  fault-tolerant measurement of $\frac{X \pm Y}{\sqrt{2}}$.

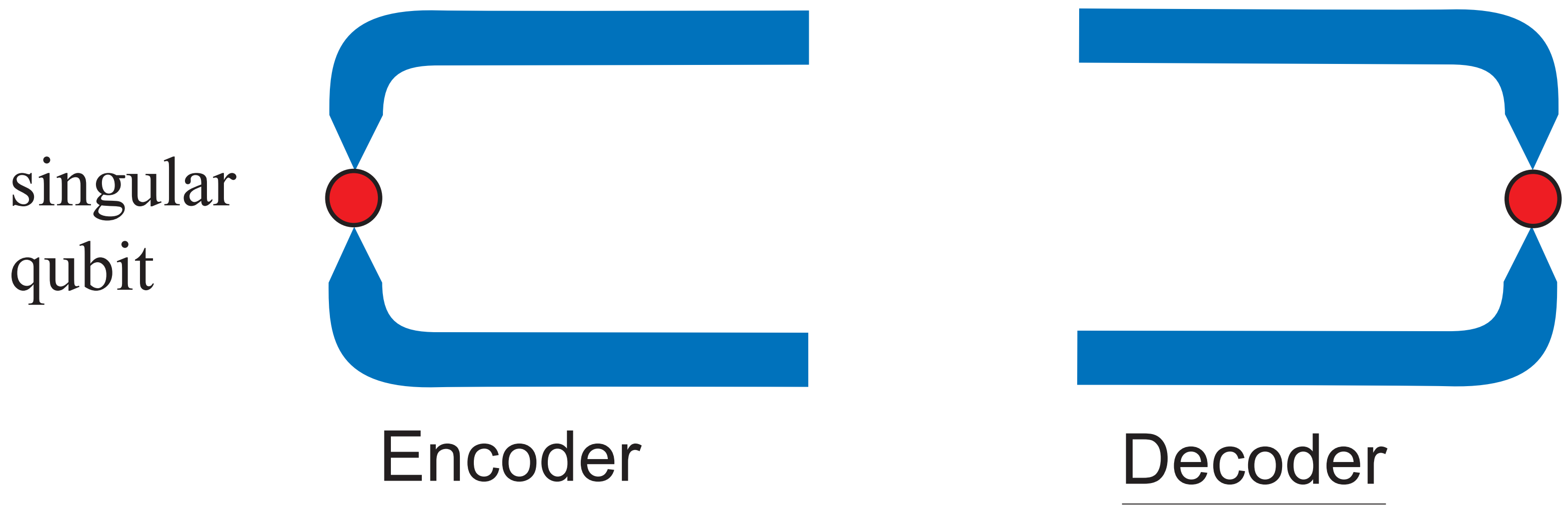
Cluster region V ✓

Defects $d \in D$ ✓

Singular qubits ←

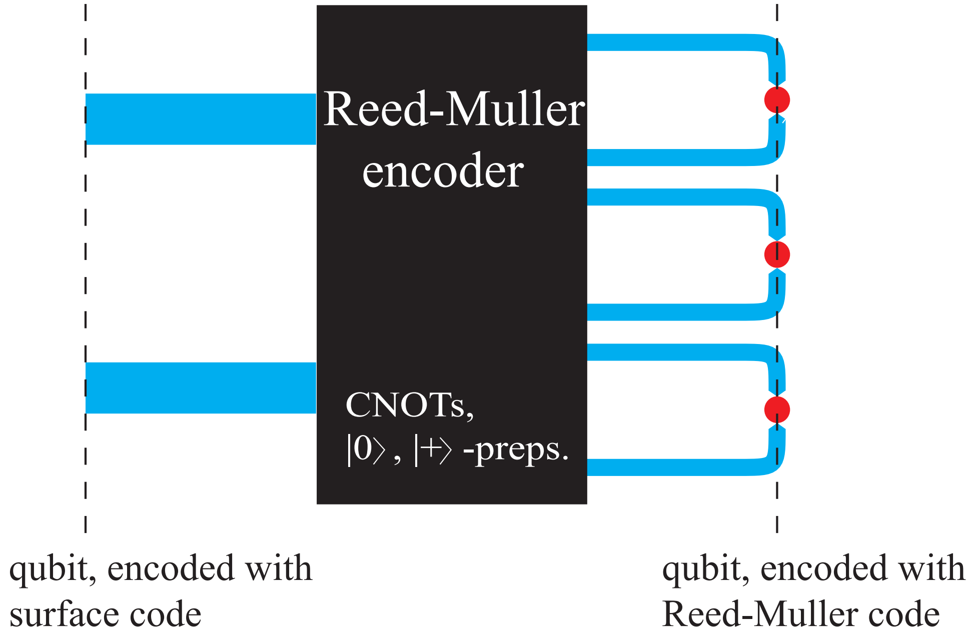
Quantum gates, Part II

Encoder and decoder for surface code:



Quantum gates, Part II

A circuit for code-conversion:



- Reed-Muller code: Fault-tolerant measurement of $\frac{\overline{X}\pm\overline{Y}}{\sqrt{2}}$ via *local* measurements of $\frac{X_a \pm Y_a}{\sqrt{2}}$ and classical post-processing.

-> Fault-tolerant universal gate set complete.

Part III:

The Error Model

Error model:

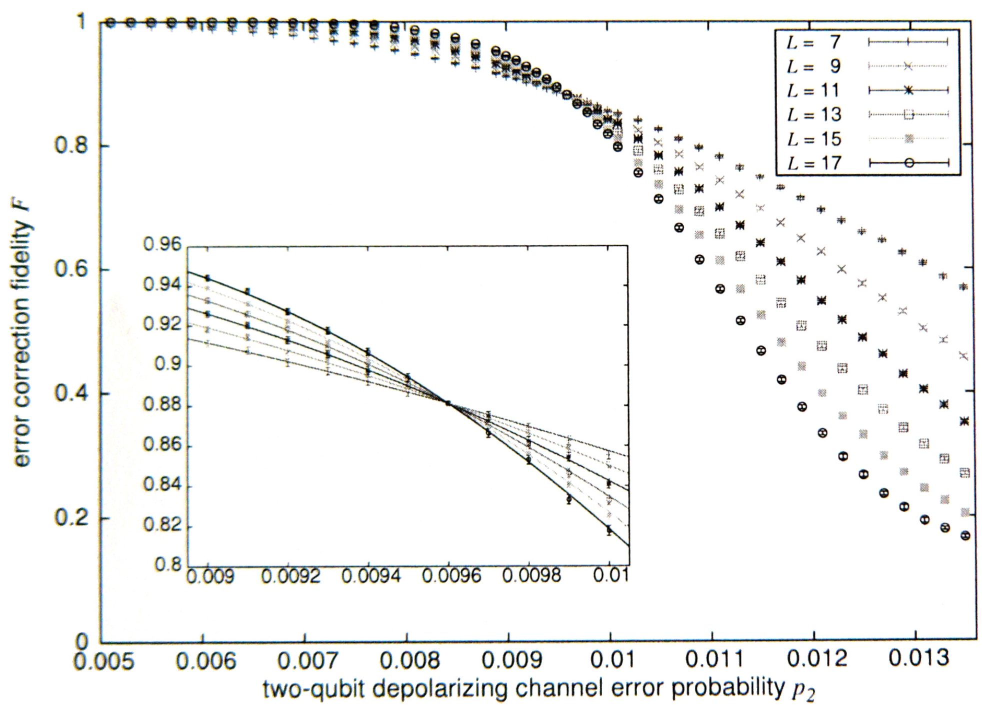
- Cluster state created in a 4-step sequence of $\Lambda(Z)$-gates from product state $\bigotimes_{a \in \mathcal{C}} |+\rangle_a$.
- Error sources:
  - $|+\rangle$-preparation: Perfect preparation followed by 1-qubit partially depolarizing noise with probability p_P.
  - $\Lambda(Z)$-gates: Perfect gates followed by 2-qubit partially depolarizing noise with probability p_2.
  - Memory: 1-qubit partially depolarizing noise with probability p_S per time step.
  - Measurement: Perfect measurement preceeded by 1-qubit partially depolarizing noise with probability p_M .
- 3D cluster state created in slices of fixed thickness.
- Instant classical processing.

Part IV:

Threshold and overhead

The fault-tolerance threshold is 1.1×10^{-3} for each source. The overhead is polynomial.

Topological error-correction in V



$$
\begin{array}{lcll}
p_{2,c} & = & 9.6 \times 10^{-3}, & \text{for } p_P = p_S = p_M = 0, \\
p_c & = & 5.8 \times 10^{-3}, & \text{for } p_P = p_S = p_M = p_2 =: p.
\end{array} \qquad (5)
$$

Reed-Muller error-correction in S

Error budget from Reed-Muller concatenation threshold:

$$\frac{76}{15}p_2 + \frac{2}{3}p_P + \frac{4}{3}p_M + \frac{4}{3}p_S < \frac{1}{105}. \quad (6)$$

Specific parameter choices:

$$\begin{aligned} p_{2,c} &= 2.9 \times 10^{-3}, \quad \text{for } p_P = p_S = p_M = 0, \\ p_c &= 1.1 \times 10^{-3}, \quad \text{for } p_P = p_S = p_M = p_2 =: p. \end{aligned} \quad (7)$$

The Reed-Muller code sets the overall threshold.

Overhead

N: Number of non-Clifford operations in bare computation.
N_{ft}: Number of operations for fault-tolerant computation.

$$N_{\text{ft}} \leq N^2 (\log N)^{10.8} . \qquad (8)$$

- Overhead is polynomial.
- Exponents may be reduced in more resouceful adaptions.

Summary

[quant-ph/0510135]

Scenario:

- Local and next-neighbor gates in 3D.

Numbers:

- Fault-tolerance threshold of 1.1×10^{-3} for preparation-, gate-, storage- and measurement error (each source).

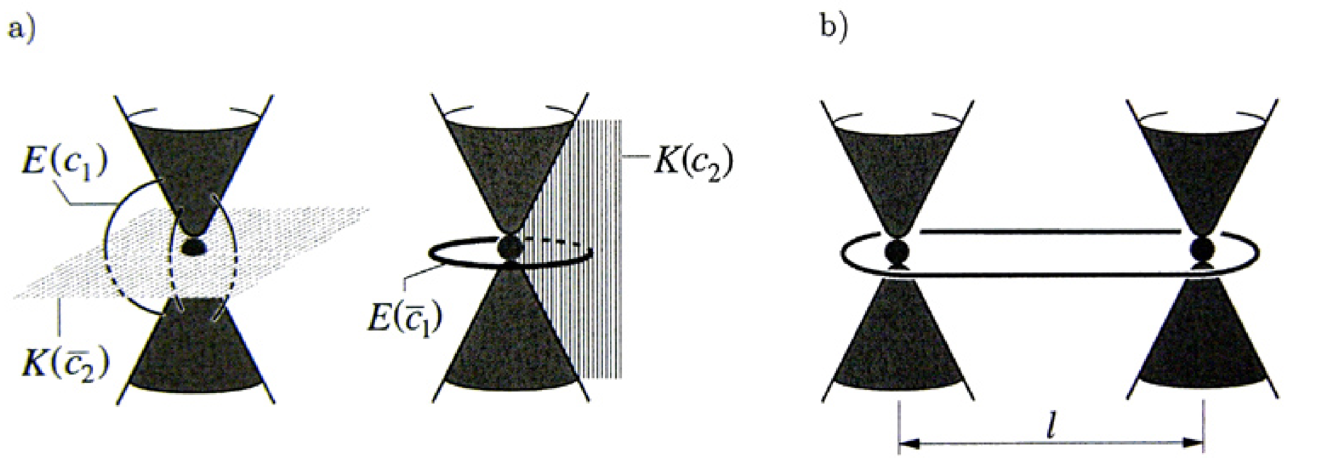
Methods:

- Cluster states in three spatial dimensions provide intrinsic topological error correction related to the *Random plaquette Z_2-gauge model*.

- Quantum logic is realized by topologically entangled *engineered lattice defects*.

Supplementary material

Local residual error on S-qubits



- Topological error correction breaks down near the S-qubits.
- Leads to *local* effective error on S-qubits.

The CNOT-gate