Simulating evolution of spin systems

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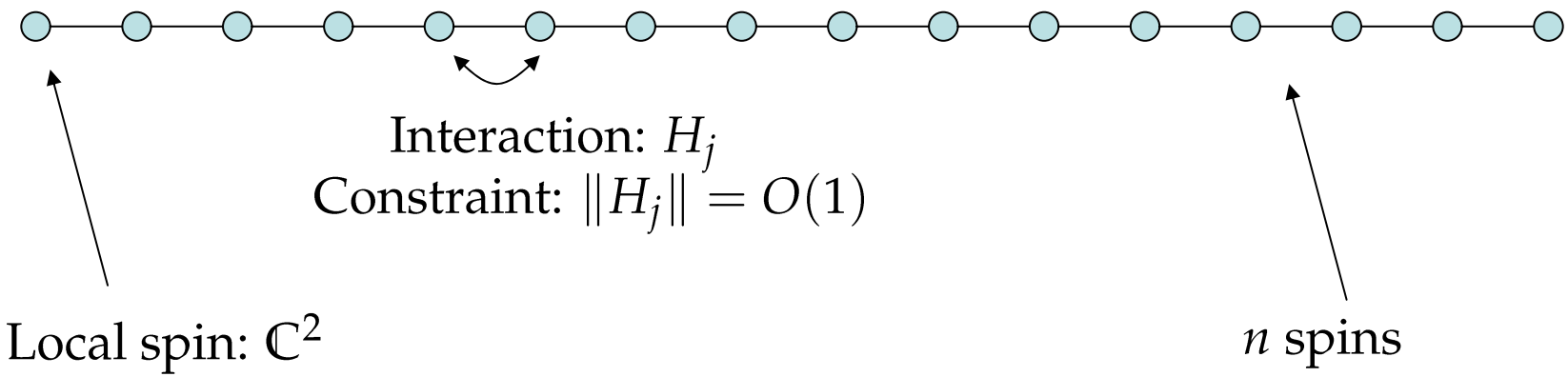
Outline

1. Introduction
2. Local answers to local questions
3. Where is the locality?
4. Let's work in the Heisenberg picture!
5. Simulating real-time evolution (proof in 1D)
6. Simulating adiabatic evolution of gapped spin systems

Introduction

We consider 1D lattices of spins which interact with their neighbours (*everything applies equally well to higher-dimensional lattices etc.*):



Hilbert space: $\mathcal{H} = (\mathbb{C}^2)^{\otimes n}$

Hamiltonian: $H = \sum_{j=0}^{n-1} H_j$

Our task: simulate the dynamics of this system for some t.

Local answers to local questions

- In condensed-matter physics we want to simulate *local quantities* such as *magnetisation* and *correlators*.
- In quantum algorithms we want to simulate the answer the algorithm produces, which should be encoded in the final state in a *local way* (so it is easy to read out).

Where is the locality?!?

Surely we can use locality to reduce cost of simulation!

But how do we talk about locality in tensor-product hilbert spaces H? Can we talk about the "locality" of states?

In what sense is a *state* on a tensor-product hilbert space "local"?

Eg. is $|01100111\rangle$ any more "local" than $\frac{1}{\sqrt{2}}(|00000000\rangle + |11111111\rangle)$?

Conclusion: we can't really talk about the locality of states in any meaningful way which connects to the computational cost of simulation. The correct ☺ way to describe locality is in the *Heisenberg picture.*

(Notable exception: in 1D one can use matrix product states (MPS) in the Schrödinger picture to efficiently obtain the expectation values of local operators. The problem is how to prove that the MPS representation can be obtained efficiently: one requires the proof of this talk in the Heisenberg picture and then one must translate the results back to the Schrödinger picture. This doesn't work in 2D! *But Heisenberg picture approach does…)*

Let's work in the Heisenberg picture!

We can certainly talk about the locality of *observables*.

Definition: a *local observable* on a subset Λ of spins is a hermitian operator M which has form $M = M_\Lambda \otimes \mathrm{I}$

Locality manifests itself when we measure M: we make a measurement of M on Λ but do *nothing* outside of Λ:

$$\langle M_\Lambda \rangle_\rho = \mathrm{tr}(M_\Lambda \rho) = \mathrm{tr}_\Lambda(M_\Lambda \rho_\Lambda)$$

Expectation values of M *don't care* about the state outside of Λ.

In the thermodynamic, or large-n, limit local observables are the **only physically accessible observables**. (Mathematically the space of local observables forms a $C^\star$-algebra: the *quasilocal algebra*.)

Heisenberg picture cont.

In the Heisenberg picture we keep track of the dynamics by *changing the observables* and leaving the initial state *unchanged*:

$$M \mapsto M(t) = e^{itH} M e^{-itH}$$

$$\rho_{\text{init}} \mapsto \rho_{\text{init}}$$

Where's the locality? We look at $M(t)$ and see how big a subset Λ of spins you have to choose so that:

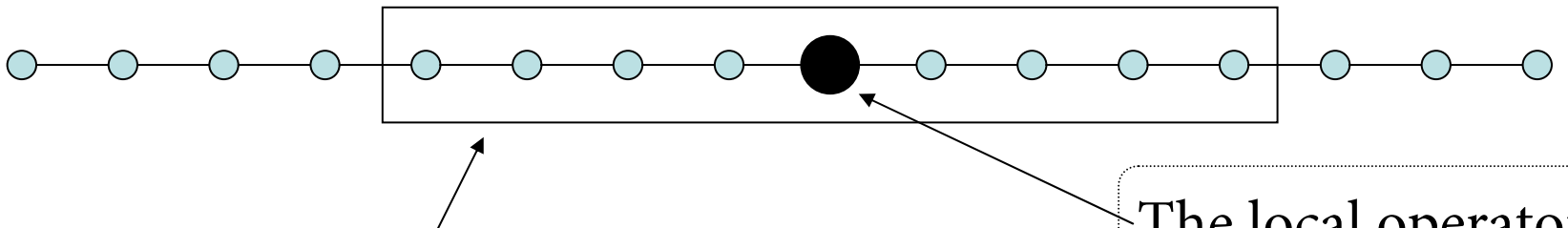
$$M(t) \approx M(t)_{\Lambda} \otimes \mathbb{I}_{\widehat{\Lambda}}$$

From now on we work in the Heisenberg picture. We assume that the initial state for our system is $|000\ldots0\rangle$. We denote expectation values with respect to this state by writing

$$\omega(M) = \langle 00\ldots0|\mathrm{M}|00\ldots0\rangle.$$

Efficiently approximating evolutions

Idea: for local operators M the evolution $M(t) = e^{itH}Me^{-itH}$ of M should be nearly the same as that with respect to H_L, the hamiltonian for a block of L contiguous spins surrounding the operator M:



The local operator M

Region L: Containing j spins

Evolution w.r.t. H_L defined by:

$$M_L(t) = \tau_t^{H_L}(M) = e^{itH_L}Me^{-itH_L}$$

where: $H_L = \sum_{j \in L} H_j$

We claim that $||M(t) - M_L(t)||$ is small when L is a big enough block around the site where M lives. ($||M||$ denotes the operator norm of M.)

How good is the approximation?

We study

$$\Delta_j(t) = \| \tau_t^{H_{L_j}}(M) - \tau_t^{H_{L_{j-1}}}(M) \|$$

where L_j is a contiguous block of j spins centred on the site where M lives.

Using fundamental theorem of calculus, unitary invariance of operator norm, and triangle inequality, we derive the following system of differential inequalities:

$$\frac{d\Delta_j(t)}{dt} \leq 2 \| H_j \| \Delta_{j-1}(t)$$

with initial conditions $\Delta_0(0) = 1$, $\Delta_j(0) = 0$, $j \neq 0$.

Solving this system by, eg., Picard iteration yields inequality

$$\Delta_j(t) \leq \frac{2^j \|h\|^j |t|^j}{j!}$$

Where $\|h\| = \max_j \|H_j\|$. This is exponentially decaying in j for constant $|t|$.

How good is the approximation cont.?

To get our final upper bound for

$$\|\tau_t^{H_{L_j}}(M) - \tau_t^H(M)\|$$

we use the triangle inequality and sum:

$$\begin{aligned}\|\tau_t^{H_{L_j}}(M) - \tau_t^H(M)\| &\leq \sum_{k=j}^{\infty} \Delta_j(t) \\ &\leq \sum_{k=j}^{\infty} \frac{2^k \|h\|^k |t|^k}{k!} \\ &\leq p(t) e^{vt-\kappa j}\end{aligned}$$

where v,κ are constants depending only on $\|h\|$, which is $O(1)$, and $p(t)$ is a lower-order polynomial in t. *Exponentially decaying in j*!

Since $\tau_t^{H_{L_j}}(M)$ can be computed with resources scaling as 2^j then we can calculate $\omega(M(t))$ efficiently for constant or logarithmic $|t|$. This can be easily extended to operators with support on a bounded number of sites.

Applications

One can easily extend these results to 2D, *and more general lattices*. With a lot more work these results can be extended to treat *adiabatic quantum evolution* of *gapped* spin systems where one obtains the following

Proposition: Let L be a regular periodic 2D lattice with m^2 sites with a quantum spin attached to each site. Let the hamiltonian $H(s)$ for this system be parameter-dependent and involve interaction terms $H_{\mathbf{j}}(s)$ labelled by the vertices of the lattice and which interact only a bounded number of spins around **j** (i.e. H(s) is a parameter-dependent local hamiltonian on a 2D lattice):

$$H = \sum_{\mathbf{j} \in L} H_{\mathbf{j}}$$

Assume $\| H_{\mathbf{j}}(s) \| < O(1)$ and that the gap $\Delta E(s)$ between ground state and 1st excited state satisfies $\Delta E(s) > O(1)$. Finally, assume that the ground state $|\Omega(0)\rangle$ of $H(0)$ known efficiently. Then the expectation values of local operators in $|\Omega(s)\rangle$ can be computed efficiently for $s < O(1)$.

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